3 Universal Behavior of Quadratic Maps

In this chapter, we study the logistic map

$$x_{n+1} = f_r(x_n) \equiv r x_n (1 - x_n) , \tag{3.1}$$

shown in Fig. 22.

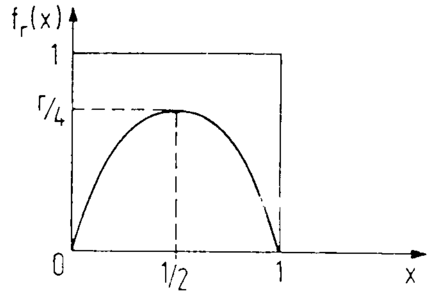

Fig. 22: The quadratic map $f_r(x)$ on the unit interval.

It has already been shown in Chapter 1 that (3.1) describes the angles x_n of a strongly damped kicked rotator. But the logistic map, which is, arguably, the simplest nonlinear difference equation, appears in many contexts.

It has already been introduced in 1845 by P. F. Verhulst to simulate the growth of a population in a closed area. The number of species x_{n+1} in the year $n + 1$ is proportional to the number in the previous year x_n and to the remaining area, which is diminished, proportionally, to x_n i.e. $x_{n+1} = r x_n (1 - x_n)$ where the parameter r depends on the fertility, the actual area of living etc.

Another example is a savings account with a self-limiting rate of interest (Peitgen and Richter, 1984). Consider a deposit z_0 which grows with a rate of interest ε as $z_{n+1} = (1 + \varepsilon) z_n = \ldots (1 + \varepsilon)^{n+1} z_0$. To prohibit unlimited wealth, some politician could suggest that the rate of interest should be reduced proportionally to z_n, i.e. $\varepsilon \to \varepsilon_0 (1 - z_n/z_{\max})$. Then the account develops according to $z_{n+1} = [1 + \varepsilon_0 (1 - z_n/z_{\max})] z_n$ which becomes equal to eq. (3.1) for $x_n = z_n \varepsilon_0 / z_{\max} (1 + \varepsilon_0)$ and $r = 1 + \varepsilon_0$.

One could expect for both examples that due to the feedback mechanism the quantities of interest (population and bank account) develop towards mean values. But as

$$-\alpha\{g'[g(0)] + 1\} = \delta\,. \tag{3.51 a}$$

The value $g'[g(0)] = g'(1)$ follows for functions with a quadratic maximum (i.e. $g''(0) \neq 0$) by differentiating the fixed-point equation (3.22) twice:

$$g''(x) = -\left\{g''\left[g\left(-\frac{x}{\alpha}\right)\right]\left[g'\left(-\frac{x}{\alpha}\right)\right]^2 + g'\left[g\left(-\frac{x}{\alpha}\right)\right]g''\left(-\frac{x}{\alpha}\right)\right\}/\alpha$$

$$\to g'(1) = -\alpha\,. \tag{3.51 b}$$

Thus (3.51 a) becomes

$$\delta = \alpha^2 - \alpha\,. \tag{3.51 c}$$

(For functions with a maximum of order $2z$ one finds $\delta = \alpha^{1+z} - \alpha$.)

Using our previously determined value $\alpha = 2.73$, we obtain $\delta \approx 4.72$ from (3.51), i.e., an accuracy of about 1% with respect to Feigenbaum's numerical result $\delta = 4.6692016\ldots$ This is not so bad if one considers the crudeness of our approximation.

It is of course much more laborious to show that δ is indeed the only eigenvalue of L_g which is larger than unity. Extensive computer calculations by Feigenbaum and the analytical results of Collet, Eckmann, and Lanford (1980) have proven this assumption.

Summarizing, the two main results of this section are

a) the fixed-point equation for the doubling operator (3.22)

$$Tg(x) = -\alpha g\left[g\left(-\frac{x}{\alpha}\right)\right] = g(x) \tag{3.52}$$

which establishes the universality of α,

b) the linearized doubling transformation (3.43)

$$T^n f_R(x) = g(x) + (R - R_\infty)\cdot\delta^n\cdot a\cdot h(x) \quad \text{for} \quad n \gg 1 \tag{3.53}$$

which shows that δ is universal and determines the way in which a function is repelled from the fixed-point function $g(x)$.

Universality emerges here because the linearized doubling operator L_g has only one *relevant* eigenvalue $\lambda_1 > 1$ such that all functions $f(x)$ – with the exception of $\varphi_1(x)$ – renormalize, after several applications of T, to the fixed-point function $g(x)$ because the eigenvalues belonging to $f - g = \sum_{\nu \neq 1} c_\nu \varphi_\nu$ are smaler than unity, i.e., *irrelevant*.

found by Grossmann and Thomae (1977), by Feigenbaum (1978), and by Coullet and Tresser (1978), and many others (see May, 1976, for earlier references) the iterates x_1, $x_2 \ldots$ of (3.1) display, as a function of the external parameter r, a rather complicated behavior that becomes chaotic at large r's (see Fig. 23).

Once can, therefore, understand the conclusion that May (1976) draws at the end of his article in "Nature":"Perhaps we would all be better off, not only in research and teaching, but also in everyday political and economical life, if more people would take into consideration that simple dynamical systems do not necessarily lead to simple dynamical behavior."

However, chaotic behavior is not tied to the special form of the logistic map. Feigenbaum has shown that the route to chaos that is found in the logistic map, the "Feigenbaum route", occurs (with certain restrictions which will be discussed below) in all first-order difference equations $x_{n+1} = f(x_n)$ in which $f(x_n)$ has (after a proper rescaling of x_n) only a single maximum in the unit interval $0 \le x_n \le 1$. It was found by Feigenbaum that the scaling behavior at the transition to chaos is governed by universal constants, the Feigenbaum constants α and δ, whose value depends only on the order of the maximum (e.g. quadratic, i.e. $f'(x_{\max}) = 0$, $f''(x_{\max}) < 0$, etc.). Because the conditions for the appearance of the Feigenbaum route are rather weak (it is practically sufficient that the Poincaré map of a system is approximately one-dimensional and has a single maximum), this route has been observed experimentally in many nonlinear systems.

The following sections of this chapter contain a rather detailed derivation of the universal properties of this route. We begin with a summary, which is intended to be a guide through the more mathematical parts.

Section 3.1 gives an overview of the numerical results for the iterates of the logistic map. It shows that the number of fixed points of $f_r(x)$ (towards which the iterates converge) doubles at distinct, increasing values of the parameter r_n. At $r = r_\infty$, the number of fixed points becomes infinite; and beyond this (finite) r-value, the behavior of the iterates is chaotic for most r's.

In Section 3.2, we investigate the pitchfork bifurcation, which provides the mechanism for the successive doubling of fixed points. It is shown that the doubling can be understood by examining the image of even iterates ($f[f(x)]$, $f[f[f[f(x)]]]$, $\ldots$) of the original map $f(x)$. This relates the generation of new fixed points to a law of functional composition. We, therefore, introduce the doubling transformation T that describes functional composition together with simultaneous rescaling along the x- and y-axis ($\mathrm{T}f(x) \equiv -\alpha f[f(-x/\alpha)]$) and show that the Feigenbaum constant α (which is related to the scaling of the distance between iterates) can be calculated from the (functional) fixed point f^* of T ($\mathrm{T}f^* = f^*$). This establishes the universal character of α. The other Feigenbaum constant δ (which measures the scaling behavior of the r_n-values) then appears as an eigenvalue of the linearized doubling transformation.

After having provided a method of calculating universal properties of the iterates, we consider several applications in Section 3.3. As a first step, we determine the relative separations of the iterates and show that the iterates form (at the accumulation point r_∞) a self-similar point set with a fractal dimensionality. We then Fourier-transform

The last equation can be generalized if we introduce the slopes

$$\mu \equiv \frac{\mathrm{d}}{\mathrm{d}x_0^*} f_r^{2^n}(x_0^*) = \prod_i f_r'(x_i^*) \tag{3.47}$$

as a parameter and characterize r by the pair (n, μ), as shown in Fig. 30.

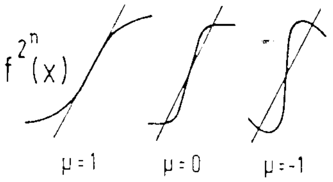

Fig. 30: Parametrization of r by n and μ (schematically), i. e. $r_n = R_{n,1} \triangleq (n, 1)$ and $R_n = R_{n,0} \triangleq (n, 0)$.

Then we obtain from (3.44):

$$\lim_{n\to\infty} (R_{n,\mu} - R_\infty) \cdot \delta^n = \frac{g_{0,\mu}(0) - g(0)}{\alpha \cdot h(0)} \tag{3.48 a}$$

where

$$g_{0,\mu}(x) = \lim_{n\to\infty} (-\alpha)^n f_{R_{n,\mu}}^{2^n}\left[\frac{x}{(-\alpha)^n}\right] \tag{3.48 b}$$

is again a universal function of μ.

At the bifurcation points, r_n, the slopes have always the same value $\mu = 1$ (see Fig. 30). Therefore, the r_n's scale according to (3.48) with the same δ as the R_n's of the superstable cycles (with $\mu = 0$):

$$r_n - r_\infty \propto \delta^{-n} \quad \text{for} \quad n \gg 1 \,. \tag{3.49 a}$$

Note that the accumulation point is the same for all μ's:

$$\lim_{n\to\infty} R_{n,\mu} = R_\infty = r_\infty \tag{3.49 b}$$

because $r_n \leq R_{n,\mu} \leq r_{n-1}$ and $r_{n+1} - r_n \to 0$ for $n \to \infty$.

The numerical value for δ can be obtained (by combining (3.35–43)) from the universal eigenvalue equation

$$\mathrm{L}_g h(x) = -\alpha \left\{ g'\left[g\left(-\frac{x}{\alpha}\right)\right] h\left(-\frac{x}{\alpha}\right) + h\left[g\left(-\frac{x}{\alpha}\right)\right]\right\} = \delta \cdot h(x) \,. \tag{3.50}$$

To make things simple we retain in the power law expansion for $h(x)$ only the first term $h(0)$ such that (3.50) becomes an algebraic equation for δ:

the distribution of iterates to obtain the experimentally measurable, and therefore important, power spectrum.

In any real dissipative nonlinear system, there are, due to the coupling to other degrees of freedom, also fluctuating forces, which when they are incorporated explicitly into the difference equations, tend to wash out the fine structure of the distribution of iterates. We determine the influence of this effect on the power spectrum and show that the rate at which higher subharmonics become suppressed scales via a power law with the noise level.

Up to this point, we have only considered the behavior of the iterates near the transition to chaos. It will be shown next that in the chaotic region ($r_\infty \le r \le 4$) periodic and chaotic r values are densely interwoven and one finds a sensitive dependence on parameter values. We also discuss the concept of structural universality and calculate the invariant density of the logistic map at $r = 4$.

Finally, in Section 3.5 we present a summary that explains the parallels between the Feigenbaum route to chaos and ordinary equilibrium second-order phase transitions. This chapter ends with a discussion of the measurable properties of the Feigenbaum route and a review of some experiments in which this route has been observed.

3.1 Parameter Dependence of the Iterates

To provide an overview in this section, we present several results for the logistic map obtained by computer iteration of eq. (3.1) for different values of the parameter r. Fig. 23 shows the accumulation points of the iterates $\{f_r^n(x_0)\}$ for $n > 300$ as a function of r together with the Liapunov exponent λ obtained via eq. (2.9).

We distinguish between a "bifurcation regime" for $1 < r < r_\infty$, where the Liapunov exponent is always negative (it becomes only zero at the bifurcation points r_n) and a "chaotic region" for $r_\infty < r \leqslant 4$, where λ is mostly positive, indicating chaotic behavior. The "chaotic regime" is interrupted by r-windows with $\lambda < 0$ where the sequence $\{f_r^n(x_0)\}$ is again periodic.

The numerical results can be summarized as follows:

1. Periodic regime

a) The values r_n, where the number of fixed points changes from 2^{n-1} to 2^n, scale like

$$r_n = r_\infty - \text{const.}\, \delta^{-n} \quad \text{for} \quad n \gg 1\,. \tag{3.2}$$

b) The distances d_n of the point in a 2^n-cycle that are closest to $x = 1/2$ (see Fig. 24) have constant ratios:

$$\frac{d_n}{d_{n+1}} = -\alpha \quad \text{for} \quad n \gg 1\,. \tag{3.3}$$

Note that L_f is only defined with respect to a function f.
Repeated application of T yields

$$T^n f_R = T^n f_{R_\infty} + (R - R_\infty) L_{T^{n-1} f_{R_\infty}} \ldots L_{f_{R_\infty}} \delta f + O[(\delta f)^2] . \quad (3.36)$$

We observe that, according to eqns. (3.18–21), $T^n f_{R_\infty}$ converges to the fixed point,

$$T^n f_{R_\infty}(x) = (-\alpha)^n f_{R_\infty}^{2^n} \left[\frac{x}{(-\alpha)^n} \right] \cong g(x) \quad \text{for} \quad n \gg 1 , \quad (3.37)$$

and (3.36) becomes approximately:

$$T^n f_R(x) \cong g(x) + (R - R_\infty) L_g^n \delta f(x) \quad \text{for} \quad n \gg 1 . \quad (3.38)$$

This equation can be further simplified if we expand $\delta f(x)$ with respect to the eigenfunctions φ_ν of L_g,

$$L_g \varphi_\nu = \lambda_\nu \varphi_\nu ; \quad \delta f = \sum_\nu c_\nu \varphi_\nu ; \quad \nu = 1, 2 \ldots \quad (3.39)$$

$$\rightarrow L_g^n \delta f = \sum_\nu c_\nu \lambda_\nu^n \varphi_\nu \quad (3.40)$$

and assume that only one of the eigenvalues λ_ν is larger than unity, i.e.,

$$\lambda_1 > 1 ; \quad |\lambda_\nu| < 1 \quad \text{for} \quad \nu \neq 1 . \quad (3.41)$$

We then obtain only the contribution from λ_1 in (3.40),

$$L_g^n \delta f \cong c_1 \lambda_1^n \varphi_1 \quad \text{for} \quad n \gg 1 , \quad (3.42)$$

and (3.38) reduces to

$$T^n f_{R_n}(x) \cong g(x) + (R - R_\infty) \cdot \delta^n \cdot a \cdot h(x) \quad \text{for} \quad n \gg 1 \quad (3.43)$$

where we introduced $c_1 \equiv a$, $\varphi_1 \equiv h$, $\lambda_1 \equiv \delta$.
The eigenvalue $\lambda_1 \equiv \delta$ is identical with Feigenbaum's constant because for $R = R_n$ and $x = 0$, (3.43) yields

$$T^n f_{R_n}(0) = g(0) + (R_n - R_\infty) \cdot \delta^n \cdot a \cdot h(0) \quad (3.44)$$

and from (3.30) we have the condition

$$T^n f_{R_n}(0) = (-\alpha)^n f_{R_n}^{2^n}(0) = 0 . \quad (3.45)$$

This leads to the desired result (note $g(0) = 1$)

$$\lim_{n \to \infty} (R_n - R_\infty) \cdot \delta^n = \frac{-1}{a \cdot h(0)} = \text{const} . \quad (3.46)$$

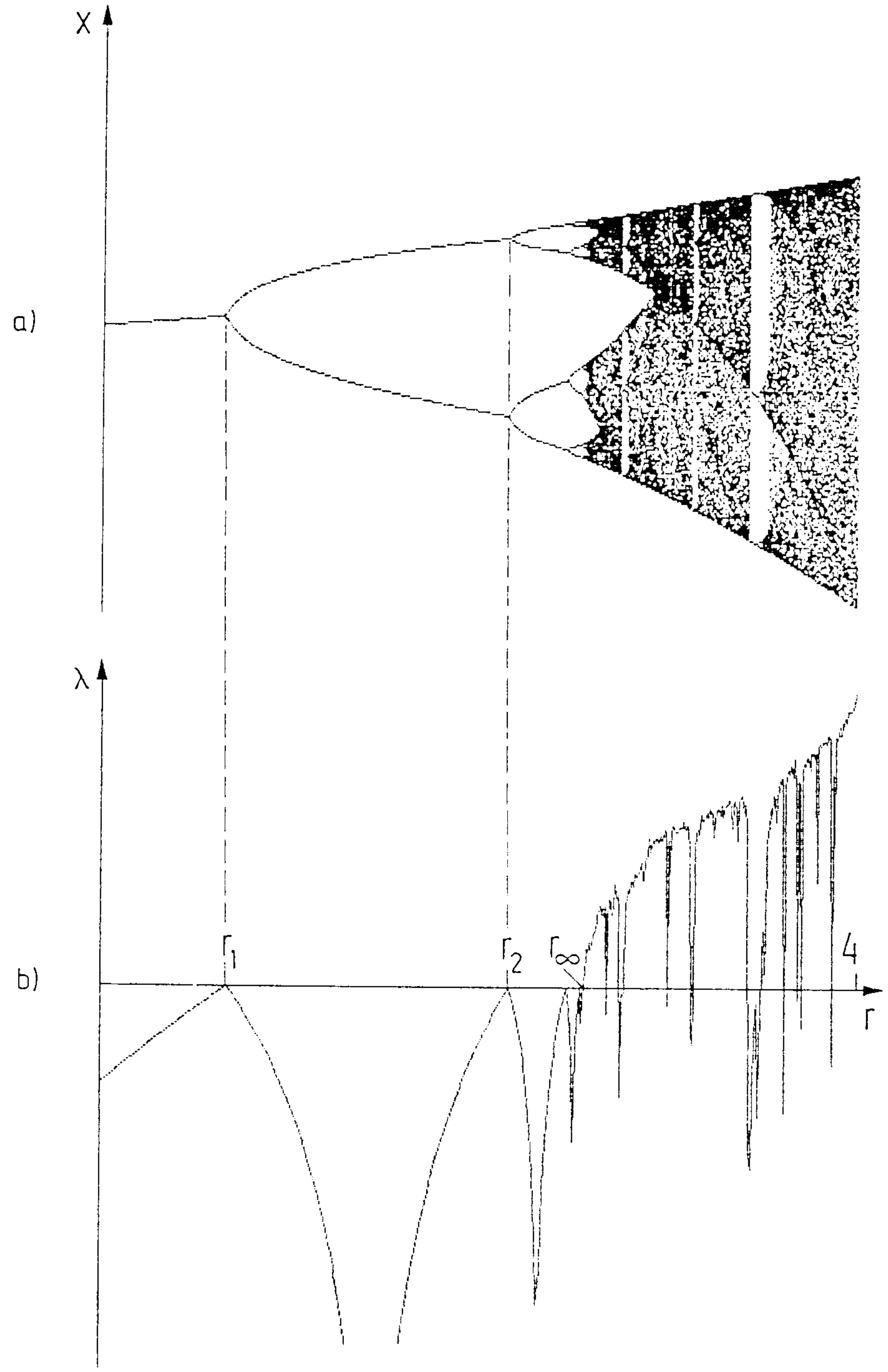

Fig. 23: a) Iterates of the logistic map, b) Liapunov exponent λ (after W. Desnizza, priv. comm.).

Linearized Doubling Transformation and δ

What can we say about the scaling along the r-axis? The values $r = R_n$, for which a 2^n-cycle becomes superstable, are determined by the condition that $x = 1/2$ is an element of the supercycle (see eq. (3.12), i.e., $x = 1/2$ is a fixed point of $f_{R_n}^{2^n}(x)$:

$$f_{R_n}^{2^n}\left(\frac{1}{2}\right) = \frac{1}{2} \tag{3.29}$$

which after translation by 1/2 becomes (see eqns. (3.13–14)):

$$f_{R_n}^{2^n}(0) = 0\,. \tag{3.30}$$

This equation has a large number of solutions because it also yields the 2^n-supercycles that occur in the windows of the chaotic regime. In order to single out the R_n-values in the bifurcation region with

$$r_1 < R_1 < r_2 < R_2 < r_3 \ldots, \tag{3.31}$$

(3.30) is solved starting from $n = 0$, and the R_n are ordered as in (3.31).

The R_n tell us how R_∞ is approached. In order to prove the scaling relation (3.5),

$$R_n - R_\infty \propto \delta^{-n}, \tag{3.32}$$

we expand $f_R(x)$ around $f_{R_\infty}(x)$:

$$f_R(x) = f_{R_\infty}(x) + (R - R_\infty)\,\delta f(x) + \ldots$$

where

$$\delta f(x) \equiv \left.\frac{\partial f_R(x)}{\partial R}\right|_{R_\infty}. \tag{3.33}$$

Let us now apply the doubling operator T to this equation. A straightforward linearization in δf yields

$$\mathrm{T} f_R = \mathrm{T} f_{R_\infty} + (R - R_\infty)\,\mathrm{L}_{f_{R_\infty}}\delta f + \mathrm{O}[(\delta f)^2] \tag{3.34}$$

where L_f is the linear operator

$$\mathrm{L}_f\,\delta f = -\alpha\left\{f'\left[f\left(-\frac{x}{\alpha}\right)\right]\delta f\left(-\frac{x}{\alpha}\right) + \delta f\left[f\left(-\frac{x}{\alpha}\right)\right]\right\}. \tag{3.35}$$

c) The Feigenbaum constants δ and α have the values

$$\delta = 4.6692016091\ldots \tag{3.4a}$$

$$\alpha = 2.5029078750\ldots \tag{3.4b}$$

Let us also note for later use that the R_n of Fig. 24 scale similar to r_n:

$$R_n - r_\infty = \text{const.}' \delta^{-n}, \tag{3.5}$$

furthermore

$$R_\infty = r_\infty = 3.5699456\ldots$$

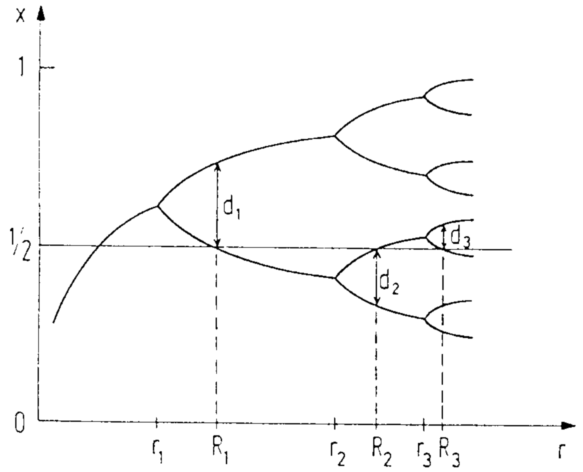

Fig. 24: Distances d_n of the fixed points closest to $x = 1/2$ for superstable 2^n-cycles (schematically).

2. Chaotic regime

a) The chaotic intervals move together by inverse bifurcations until the iterates become distributed over the whole interval [0, 1] at $r = 4$.

b) The r-windows are characterized by periodic p-cycles ($p = 3, 5, 6 \ldots$) with successive bifurcations p, $p \cdot 2^1$, $p \cdot 2^2$ etc. The corresponding r-values scale like (3.2) with the same δ but different constants.

c) Also, period triplings $p \cdot 3^n$ and quadruplings $p \cdot 4^n$, etc. occur at $\bar{r} = \bar{r}_\infty - \text{const.}\, \bar{\delta}^{-n}$ with different Feigenbaum constants $\bar{\delta}$, which are again universal (e.g. $\bar{\delta} = 55.247\ldots$ for $p \cdot 3^n$).

3.2 Pitchfork Bifurcation and the Doubling Transformation

In this section, we show that the "Feigenbaum route" in Fig. 23 is generated by pitchfork bifurcations that relate the emergence of new branches to a universal law of functional composition. By introducing the doubling transformation T (which describes

By taking the limit $i \to \infty$ in (3.19), the function

$$g(x) \equiv \lim_{i\to\infty} g_i(x) \tag{3.21}$$

becomes a fixed point of the doubling operator T:

$$g(x) = \mathrm{T}g(x) = -\alpha g\left[g\left(-\frac{x}{\alpha}\right)\right]. \tag{3.22}$$

This equation determines α universally by

$$g(0) = -\alpha g[g(0)]. \tag{3.23}$$

It can easily be shown that $\mu g(x/\mu)$ is also a solution of the fixed-point equation (3.22) with the same α. Thus, the theory has nothing to say about absolute scales, and we fix μ by setting

$$g(0) = 1. \tag{3.24}$$

Although a general theory for the solution of the functional equation (3.22) is still lacking, we can obtain a unique solution if we specify the nature of the maximum of $g(x)$ at $x = 0$ (for example quadratic) and require that $g(x)$ is a smooth function. If we use for $g(x)$ in the quadratic case the extremely short power law expansion

$$g(x) = 1 + bx^2 \tag{3.25}$$

the fixed point equation (3.22) becomes

$$1 + bx^2 = -\alpha(1 + b) - \left(\frac{2b^2}{\alpha}\right)x^2 + O(x^4) \tag{3.26}$$

which yields

$$b = (-2 - \sqrt{12})/4 \simeq -1.366; \quad \alpha = |2b| \simeq 2.73. \tag{3.27}$$

These values only differ by 10% from Feigenbaum's numerical results

$$\begin{aligned} g(x) &= 1 - 1.52763x^2 + 0.104815x^4 + 0.0267057x^6 - \dots \\ \alpha &= 2.502807876\dots \end{aligned} \tag{3.28}$$

This establishes the universality of α.

this law), we show that the Feigenbaum constants α and δ are indeed universal. They appear as the (negative inverse) value of the eigenfunction of T at $x = 1$ and as the only relevant eigenvalue of the linearized doubling operator, respectively.

Pitchfork Bifurcations

As a first step, we investigate the stability of the fixed points of $f_r(x)$ and $f_r^2(x) = f_r[f_r(x)]$ as a function of r. Fig. 25 shows that $f_r(x)$ has, for $r < 1$, only one stable fixed point at zero, which becomes unstable for $1 < r < 3$ in favor of $x^* = 1 - 1/r$.

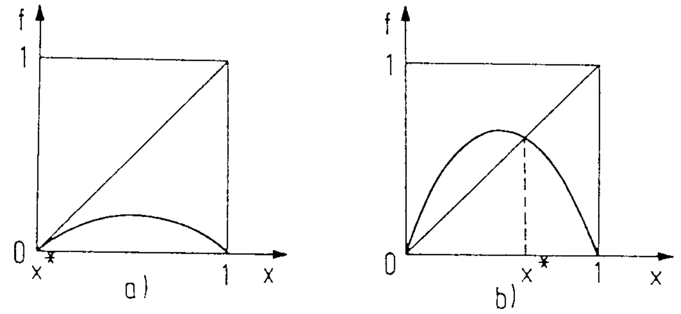

Fig. 25: The fixed points of f_r for a) $r < 1$ and b) $1 < r < 3$.

For $r > 3 = r_1$ we have $|f_r'(x^*)| = |2 - r| > 1$; i.e., x^* also becomes unstable according to criterion (2.17). What happens then?

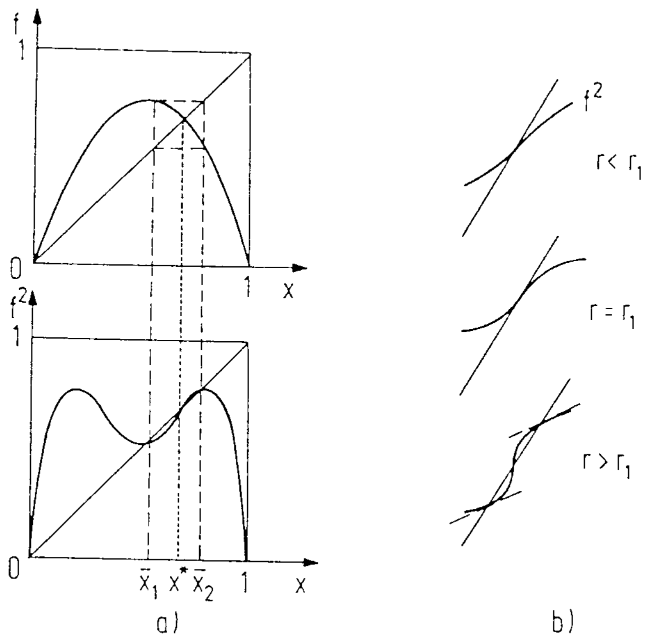

Fig. 26: a) $f(x)$ and $f^2(x) = f[f(x)]$ for $r > r_1$. b) Generation of two new stable fixed points in f^2 via a pitchfork bifurcation. (The bifurcation diagram looks like a pitchfork, see p. 182.)

Fig. 26 shows $f_r(x)$ together with its second iterate $f_r^2(x)$ for $r > r_1$. We note four properties of f^2 (the index r is dropped for convenience):

From the previous section, we see that eq. (3.3) implies

$$\lim_{n\to\infty} (-\alpha)^n d_{n+1} = d_1 \tag{3.15}$$

i.e. the sequence of scaled iterates $f^{2^n}_{\bar{R}_{n+1}}(0)$ converges:

$$\lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{R}_{n+1}}(0) = d_1 \,. \tag{3.16}$$

Fig. 29 suggests that (3.16) can be generalized to the whole interval, and the rescaled functions $(-\alpha)^n f^{2^n}_{\bar{R}_{n+1}}[x/(-\alpha)^n]$ converge to a limiting function $g_1(x)$:

$$\lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{R}_{n+1}}\left[\frac{x}{(-\alpha)^n}\right] = g_1(x) \tag{3.17}$$

Eq. (3.17) shows that $g_1(x)$ is determined only by the behavior of $f^{2^n}_{\bar{R}_{n+1}}$ around $x = 0$ (see also Fig. 28) and should, therefore, be universal for all functions f with a quadratic maximum.

Doubling Transformation and α

As the next step, we introduce, by analogy to eq. (3.17), a whole family of functions

$$g_i(x) \equiv \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{R}_{n+i}}\left[\frac{x}{(-\alpha)^n}\right]; \quad i = 0, 1 \ldots \tag{3.18}$$

We notice that all these functions are related by the *doubling transformation* T:

$$g_{i-1}(x) = (-\alpha)\, g_i\left[g_i\left(-\frac{x}{\alpha}\right)\right] \equiv \mathrm{T}\, g_i(x) \tag{3.19}$$

because

$$\begin{aligned}
g_{i-1}(x) &= \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{\bar{R}_{n+i-1}}\left[\frac{x}{(-\alpha)^n}\right] \\
&= \lim_{n\to\infty} (-\alpha)(-\alpha)^{n-1} f^{2^{n-1+1}}_{\bar{R}_{n-1+i}}\left[-\frac{1}{\alpha}\frac{x}{(-\alpha)^{n-1}}\right] \\
&= \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^m}_{\bar{R}_{m+i}}\left\{\frac{1}{(-\alpha)^m}(-\alpha)^m f^{2^m}_{\bar{R}_{m+i}}\left[-\frac{1}{\alpha}\frac{x}{(-\alpha)^m}\right]\right\} \\
&= -\alpha\, g_i\left[g_i\left(-\frac{x}{\alpha}\right)\right].
\end{aligned} \tag{3.20}$$

a) It has three extrema with $f^{2\prime} = f'[fx)]f'(x) = 0$ at $x_0 = 1/2$, because $f'(1/2) = 0$, and at $x_{1,2} = f^{-1}(1/2)$, because $f'[f[f^{-1}(1/2)]] = f'(1/2) = 0$.

b) A fixed point x^* of $f(x)$ is also a fixed point of $f^2(x)$ (and all higher iterates).

c) If a fixed point x^* becomes unstable with respect to $f(x)$, it becomes also unstable with respect to f^2 (and all higher iterates) because $|f'(x^*)| > 1$ implies $|f^{2\prime}(x^*)| = |f'[f(x^*)]f'(x^*)| = |f'(x^*)|^2 > 1$.

d) For $r > 3$, the old fixed point x^* in f^2 becomes unstable, and two new stable fixed points $\bar{x}_1, \bar{x}_2$ are created by a pitchfork bifurcation (see Fig. 26b).

The pair $\bar{x}_1, \bar{x}_2$ of stable fixed points of f^2 is called an attractor of $f(x)$ of period two because any sequence of iterates which starts in [0, 1] becomes attracted by $\bar{x}_1, \bar{x}_2$ in an oscillating fashion as shown in Fig. 27.

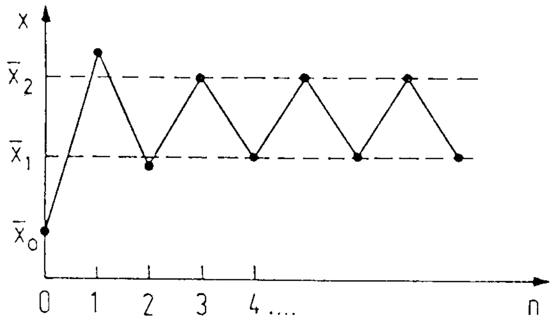

Fig. 27: Iterates of $\bar{x}_0$ if $f(x)$ has an attractor of period two (schematically).

It is easy to see that $f(x)$ maps these new fixed points of f^2 onto each other, i.e.,

$$f(\bar{x}_1) = \bar{x}_2 \text{ and } f(\bar{x}_2) = \bar{x}_1 \tag{3.6}$$

because $f^2(\bar{x}_1) = \bar{x}_1$ implies

$$ff[f(\bar{x}_1)] = f[f^2((\bar{x}_1)] = f(\bar{x}_1) \tag{3.7}$$

i.e. $f(\bar{x}_1)$ is also a fixed point of f^2, and $\bar{x}_2$ is the only possible choice. ($f(\bar{x}_1) = 0$ or x^* are at variance with $ff(\bar{x}_1) = \bar{x}_1$.)

If we now increase r beyond a value r_2, the fixed points of f^2 also become unstable. Because the derivative is the same at $\bar{x}_1$ and $\bar{x}_2$

$$f^{2\prime}(\bar{x}_1) = f'[f(\bar{x}_1)]f'(\bar{x}_1) = f'(\bar{x}_2)f'(\bar{x}_1) = f^{2\prime}(\bar{x}_2) \tag{3.8}$$

they even become unstable simultaneously.

Fig. 28 shows that after this instability the fourth iterate $f^4 = f^2 \cdot f^2$ displays two more pitchfork bifurcations which lead to an attractor of period four; i.e., one observes *period doubling*. These two examples can be generalized as follows:

which implies that it always contains $x_0^* = 1/2$ as a cycle element because this is the only point where $f_r' = 0$. Referring to Fig. 24, we can see that the distances d_n are just the distances between the cycle elements $x^* = 1/2$ and $x_1 = f_{R_n}^{2^{n-1}}(1/2)$, i.e.,

$$d_n = f_{R_n}^{2^{n-1}}\left(\frac{1}{2}\right) - \frac{1}{2} . \tag{3.13}$$

In the following it is convenient to perform a coordinate transformation that displaces $x = 1/2$ to $x = 0$ such that (3.13) becomes

$$d_n = f_{R_n}^{2^{n-1}}(0) . \tag{3.14}$$

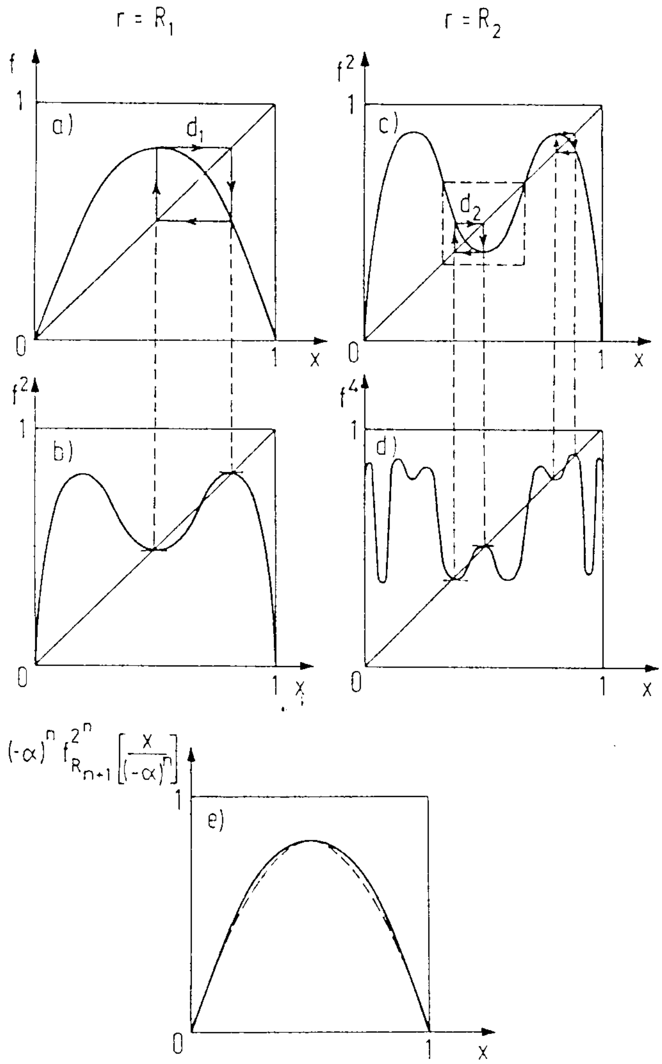

Fig. 29: The rescaled iterates $f_{R_{n-1}}^{2^n}(x)$ converge towards a universal function. a)–d) Superstable cycles at R_1 and R_2. Note the horizontal tangents in b) and d). e) The content of the dashed square of c) is rescaled (dashed line) and compared to the whole of a) (full line).

a) For $r_{n-1} < r < r_n$, there exists a stable 2^{n-1}-cycle with elements $x_0^*, x_1^* \ldots x_{2^{n-1}-1}^*$ that is characterized by

$$f_r(x_i^*) = x_{i+1}^*, \quad f_r^{2^{n-1}}(x_i^*) = x_i^*, \quad \left|\frac{d}{dx_0^*} f_r^{2^{n-1}}(x_0^*)\right| = \left|\prod_i f_r'(x_i^*)\right| < 1 \tag{3.10}$$

b) At r_n, all points of the 2^{n-1}-cycle become unstable simultaneoulsy via pitchfork bifurcations in

$$f_r^{2^n} = f_r^{2^{n-1}} \cdot f_r^{2^{n-1}} \tag{3.11}$$

that, for $r_n < r < r_{n+1}$, lead to a new stable 2^n-cycle.

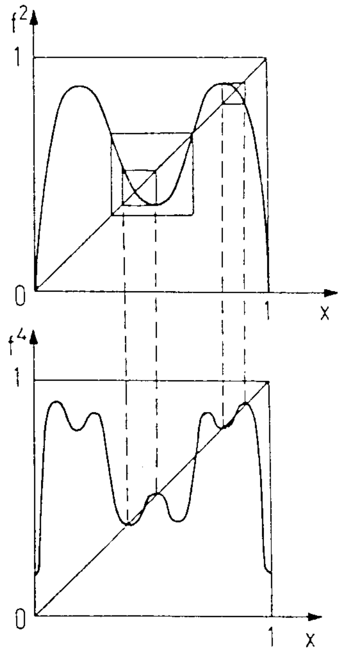

Fig. 28: Two pitchfork bifurcations in f^4 lead to an attractor of period 4.

Our last conclusion represents a first step towards universality because it connects the mechanism of subsequent bifurcations to a general law of functional composition.

Let us add as a caveat that not all quadratic maps of the unit interval onto itself display an infinite sequence of pitchfork bifurcations, but only those which have a negative Schwarzian derivative (see Appendix C).

Supercycles

To progress further, we now consider the so-called supercycles. A 2^n-supercycle is simply a superstable 2^n-cycle defined by

$$\frac{d}{dx_0^*} f_{R_n}^{2^n}(x_0^*) = \prod_i f_{R_n}'(x_i^*) = 0 \tag{3.12}$$