§7. Julia sets and the Mandelbrot set

In this section, we will explore the logistic map in the complex plane. This topic may not be as applied as the real logistic map. But the stunning beauty and striking complexity of the Julia sets and the Mandelbrot set that arise make this topic a feature attraction in the garden of mathematics. Here mathematics and art are mixed, and their distinctions blurred.

For the convenience of our discussions, we first rewrite the logistic map as

$$f_c(z) = z^2 + c \tag{7.1}$$

where c is a complex parameter. Actually it is easy to see that this quadratic map and the logistic map $g(w) = \lambda w(1 - w)$ are topologically conjugate under the transformation $w = \frac{1}{2} - \frac{z}{\lambda}$ and $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$.

Next we study the long-time behaviors of the orbits in the complex plane under the map (7.1). With the memory of the real logistic map afresh, you may be less inclined to predict simple dynamics in this map, even though it looks indeed simple. This attitude turns out to be shrewd. In fact, what you will see next may be beyond your wildesst dreams.

First we introduce some notations. For a given value of c and a starting point z, either $f_c^n(z) \nrightarrow \infty$ or remains bounded for all n.

Let $k_c = \{z : f_c^n(z) \nrightarrow \infty\}$.
It is called the filled-in Julia set.
The set $J_c = \partial k_c$ is called the Julia set.

How do the Julia set look like for various c values?

Let's first explore this on a computer. What we find is given next.

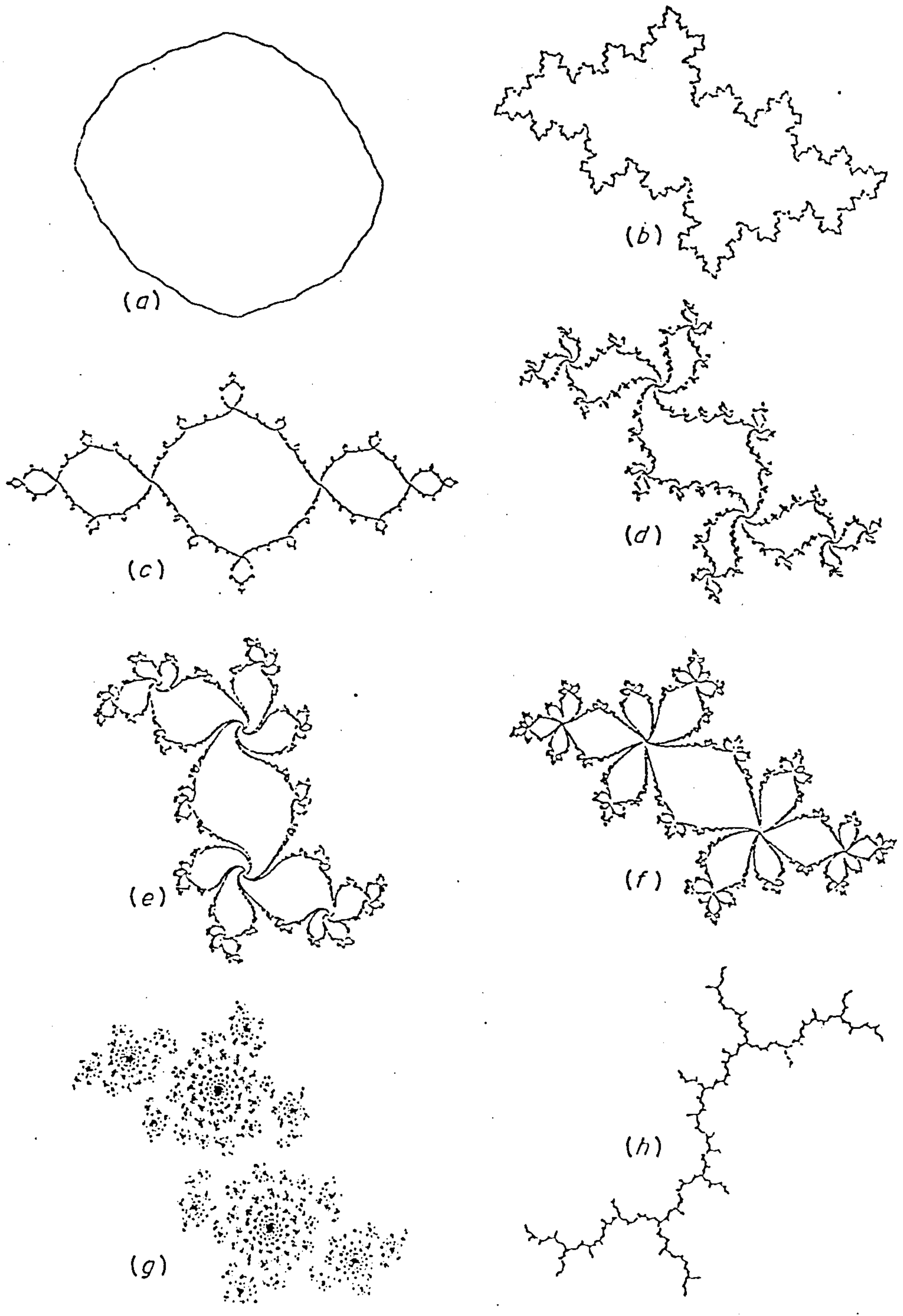


Figure 14.7 A selection of Julia sets of the quadratic function $f_c(z) = z^2 + c$. (*a*) $c = -0.1 + 0.1i$; f_c has an attractive fixed point, and J is a quasi-circle. (*b*) $c = -0.5 + 0.5i$; f_c has an attractive fixed point, and J is a quasi-circle. (*c*) $c = -1 + 0.05i$; f_c has an attractive period-2 orbit. (*d*) $c = -0.2 + 0.75i$, f_c has an attractive period-3 orbit. (*e*) $c = 0.25 + 0.52i$; f_c has an attractive period-4 orbit. (*f*) $c = -0.5 + 0.55i$; f_c has an attractive period-5 orbit. (*g*) $c = 0.66i$; f_c has no attractive orbits and J is totally disconnected. (*h*) $c = -i$, $f_c^2(0)$ is periodic and J is a dendrite

Observations:

1. Quasi-self-similarity: If you blow up a small portion of the Julia set, what you get is roughly a copy of the whole set at a reduced scale. Refer to computer images to be shown in class.
2. Fractal geometry: The Julia sets are usually fractals. In fact, they are always fractals except the two cases where $c = 0$ and -2.
3. Connectivity: The Julia sets sometimes are connected, and sometimes are not.

How can we understand and predict these exotic structures of the Julia sets for a given value of c? How can we classify them?

The key lies in the orbit behavior of $z = 0$ under this map. Why is that? There are at least two reasons.

(1) $z = 0$ is the only critical point of the map (7.1). It is known that if map (7.1) has an attracting periodic orbit, then the orbit of $z = 0$ must be attracted to this periodic orbit. Thus the orbit of 0 can track down the attracting cycle if there is one.

(2) It has been shown that

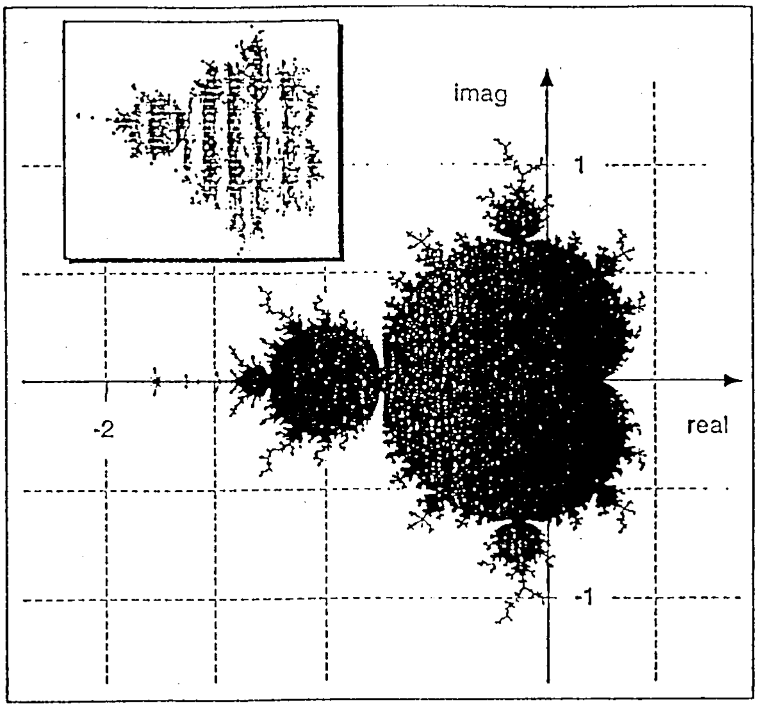
the Julia set J is connected $\Leftrightarrow$ the orbit $\{f_c^n(0)\}$ is bounded,
i.e. $f_c^n(0) \nrightarrow \infty$ as $n \to \infty$.

Thus the orbit of $z = 0$ is intimately related to the connectivity of the Julia set.

In view of the importance of the orbit of zero, we define the parameter set

$$M = \{c : f_c^n(0) \nrightarrow \infty\}$$

which is shown below.



The Mandelbrot Set — Old and New Rendering

The insert shows an original print-out from Mandelbrot's experiment. We have produced the large Mandelbrot set using a modern laser printer and a more accurate mathematical algorithm.

Figure 14.3

This is the celebrated Mandelbrot set, discovered first by B Mandelbrot in 1980. This set contains an enormous amount of information about the structure of Julia sets. Simply speaking, it is the road map or table of contents for Julia sets.

Observations:

1. The set M has a main cardioid and countably infinite buds attached to it. Let us call them primary buds. Each primary bud is decorated by countably infinite smaller buds on its boundary, which we will call as secondary buds, and so on. This is not all. In addition, fine, branched "antennas" grow outwards from the buds, and these antennas carry miniature copies of the entire Mandelbrot set along their length. The complexity of the set M can be better appreciated from the graphs on the next page, which are a series of zooms into M.

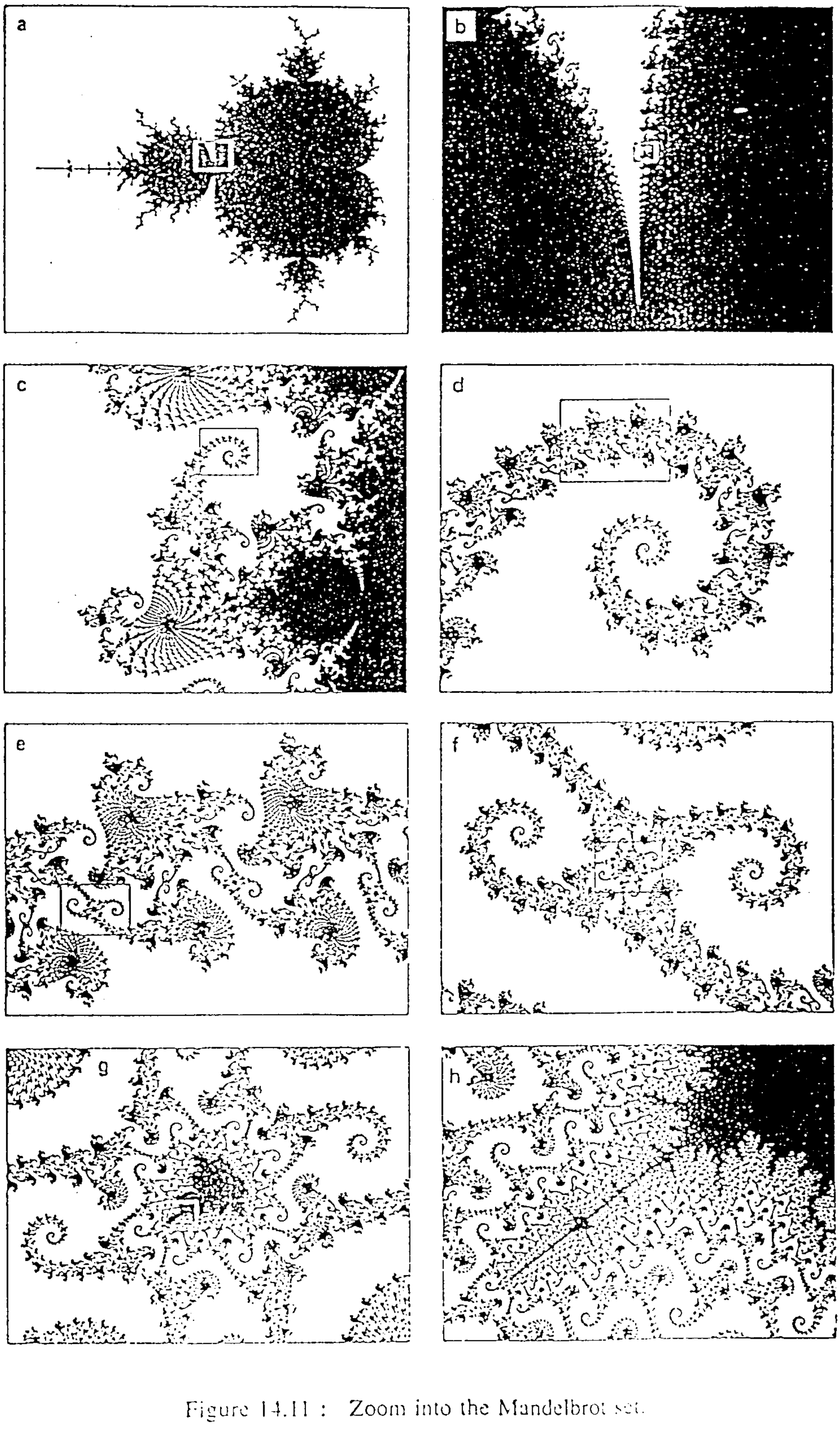


Figure 14.11 : Zoom into the Mandelbrot set.

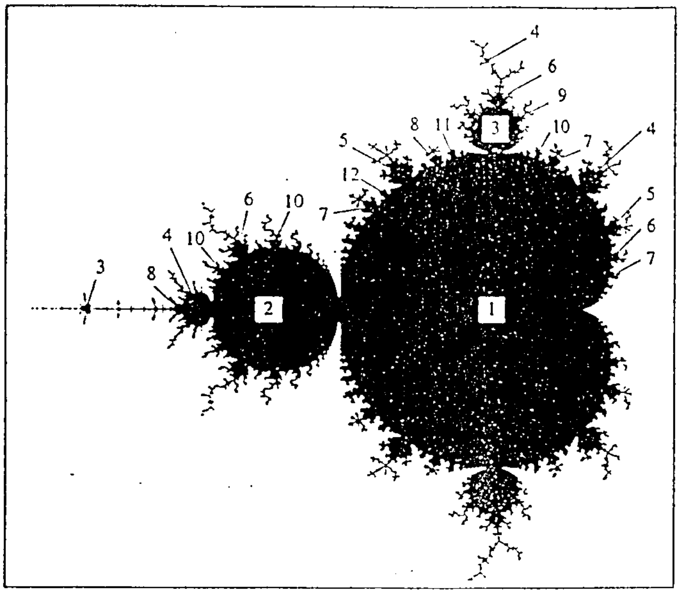
2. When c is in the main cardioid, $f_c^n(0)$ approaches an attracting fixed point and the Julia set is a quasi-circle. When c is in the buds of M, $f_c^n(0)$ approaches an attracting periodic orbit with period the same throughout each bud. The period p is shown below for many of the prominent buds in M. Note that the Mandelbrot set is symmetric about the real (z) axis. Thus the buds below this axis have the same periods as their symmetric counterparts above.

Note: The periods of these buds satisfy the Fibanacci property. For example, on the boundary of the main cardioid, the largest bud between the period–2 and 3 buds has period 5. Similarly, the largest bud in between the period–3 and 5 ones has period 8, etc. This property also applies to secondary and higher-degree buds in M.

3. If c is in a primary period–p bud, then there are p arms meeting at a point in the Julia set J_c. If c is in secondary or higher-degree buds, its Julia set will be different.

4. When c is outside of M, there are no attracting periodic orbits, and $f_c^n(0) \to \infty$. The Julia set is then a totally disconnected Cantor dust.

The structures of J_c for various values of c are displayed in Fig 14.6 on the next page.



The Mandelbrot Set and Its Atoms

The buds of the Mandelbrot set correspond to Julia sets that bound basins of attraction of periodic orbits. The numbers in the figure indicate the periods of these orbits.

Figure 14.19

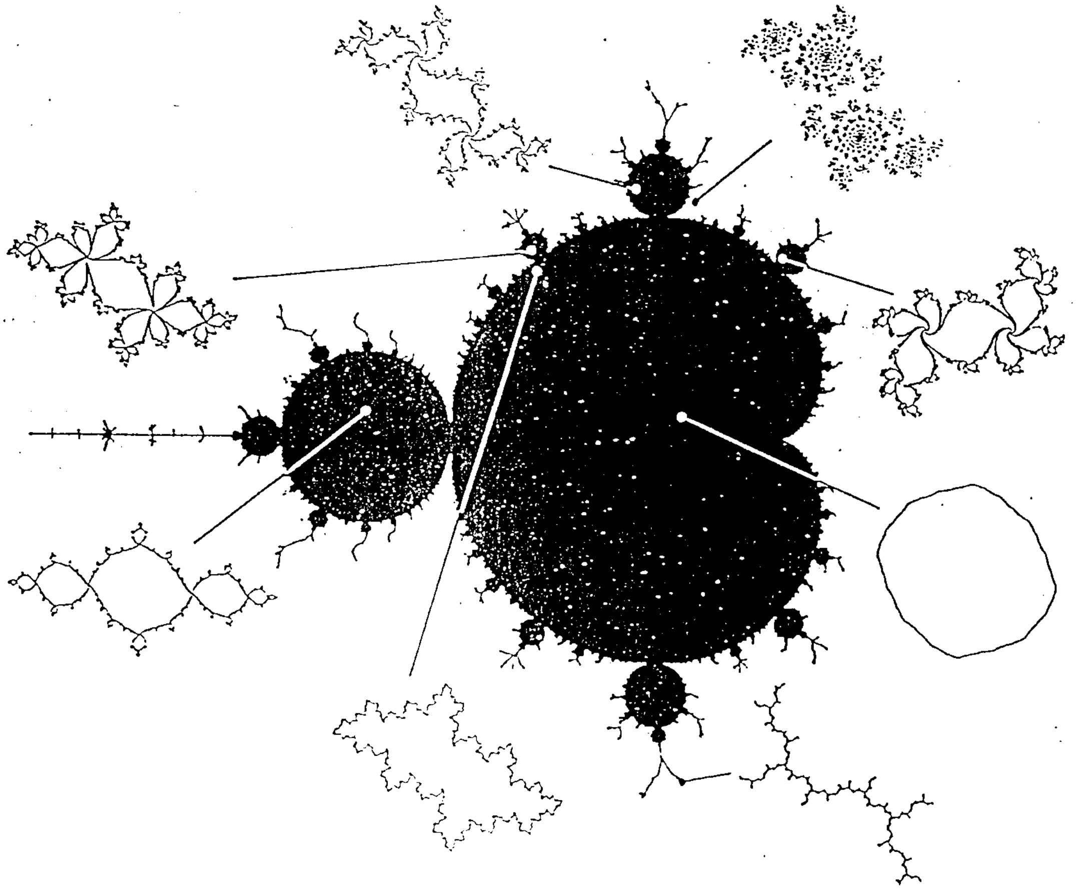


Figure 14.6 Julia sets $J(f_c)$ for c at various points in the Mandelbrot set. The Julia sets are displayed in more detail in figure 14.7

5. On the antennas of the Mandelbrot set there are infinite miniature copies of the entire set M which are all connected to each other. To substantiate this claim, look at the following graph which relates the Mandelbrot set to the bifurcation diagram of the real map $f(x) = x^2 + c$, where c is a real number.

Final-State Diagram and Mandelbrot Set

The final-state diagram of $z \to z^2 + c$ in comparison to the Mandelbrot set.

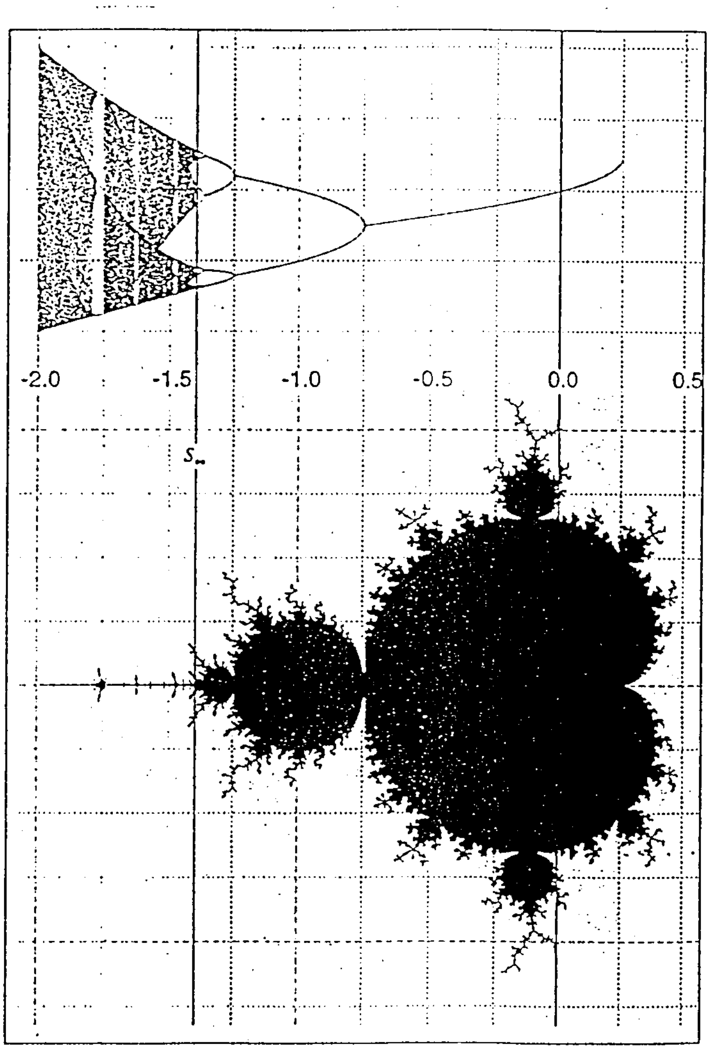


Figure 14.27

Observe that each periodic window in the real map corresponds to a tiny copy of the Mandelbrot set located on the negative x-axis. Since there are infinite period windows in the real map, infinite little Mandelbrot bugs can be found along the x-axis.

When $-2 \le c \le \frac{1}{4}$, $f_c : [-\frac{r}{2}, \frac{r}{2}] \to [-\frac{r}{2}, \frac{r}{2}]$ where $c = \frac{r}{2} - \frac{r^2}{4}$ and $1 \le r \le 4$.

Thus $\{f_c^n(0)\}$ is bounded $\Rightarrow c \in M$. Hence all those tiny copies of the Mandelbrot set on the x-axis are connected by the x interval $[-2, \frac{1}{4}]$.