Introduction to the Hirota bilinear method

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Abstract

We give an elementary introduction to Hirota's direct method of constructing multisoliton solutions to integrable nonlinear evolution equations. We discuss in detail how this works for equations in the Korteweg–de Vries class. We also show how Hirota's method can be used to search for new integrable evolution equations and list the results that have been obtained before for the mKdV/sG and nlS classes.

1 Why the bilinear form ?

In 1971 Hirota introduced a new direct method for constructing multisoliton solutions to integrable nonlinear evolution equations [1]. The idea was to make a transformation into new variables, so that in these new variables multisoliton solutions appear in a particularly simple form. The method turned out to be very effective and was quickly shown to give N-soliton solutions to the Korteweg–de Vries (KdV) [1], modified Korteweg–de Vries (mKdV) [2], sine-Gordon (sG) [3] and nonlinear Schrödinger (nlS) [4] equations. It is also useful in constructing their Bäcklund transformations [5]. Later it was observed that the new dependent variables (called "τ-functions") have very good properties and this has become a starting point for further developments.

In this talk our aim is to describe how multisoliton solutions can be constructed using Hirota's method. Multisoliton solutions can, of course, be derived by many other methods, e.g., by the inverse scattering transform (IST) and various dressing methods. The advantage of Hirota's method over the others is that it is algebraic rather than analytic. The IST method is more powerful (it can handle general initial conditions) and at the same time more complicated. Accordingly, if one just wants to find soliton solutions, Hirota's method is the fastest in producing results.

2 From nonlinear to bilinear

The (integrable) PDE's that appear in some particular (physical) problem are not usually in the best form for further analysis. For constructing soliton solutions the best form is Hirota's bilinear form (discussed below) and soliton solutions appear as polynomials of simple exponentials only in the corresponding new variables. The first problem we face is therefore to find the bilinearizing transformation. This is not algorithmic and can sometimes require the introduction of new dependent and sometimes even independent variables.

Here we will discuss in detail only the KdV equation

$$u_{xxx} + 6uu_x + u_t = 0. \tag{1}$$

Since we have not yet defined what bilinear is, let us first concentrate on transforming the equation into a form that is quadratic in the dependent variables. One guideline in searching for the transformation is that the leading derivative should go together with the nonlinear term, and, in particular, have the same number of derivatives. If we count a derivative with respect to x having degree 1, then to balance the first two terms of (1) u should have degree 2. Thus we introduce the transformation to a new dependent variable w (having degree 0) by

$$u = \partial_x^2 w. \tag{2}$$

After this the KdV equation can be written

$$w_{xxxxx} + 6w_{xx}w_{xxx} + w_{xxt} = 0, \tag{3}$$

which can be integrated once with respect to x to give

$$w_{xxxx} + 3w_{xx}^2 + w_{xt} = 0. \tag{4}$$

In principle this would introduce an integration constant (function of t), but since (2) defines w only up to $w \to w + x\lambda(t)$, we can use this freedom to absorb it.

Equations of this form can usually be bilinearized by introducing a new dependent variable whose natural degree (in the above sense) would be zero, e.g., $\log F$ or f/g. In this case the first one works, so let us define

$$w = \alpha \log F \tag{5}$$

with a free parameter α. This results in an equation that is fourth degree in F, with the structure

$$F^2 \times (\text{something quadratic}) + 3\alpha(2-\alpha)(2FF'' - F'^2)F'^2 = 0. \tag{6}$$

Thus we get a quadratic equation if we choose $\alpha = 2$, and the result is

$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0. \tag{7}$$

In addition to being quadratic in the dependent variables, an equation in the Hirota bilinear form must also satisfy a condition with respect to the derivatives: they should only appear in combinations that can be expressed using Hirota's D-operator, which is defined by:

$$D_x^n \, f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x}. \tag{8}$$

Thus D operates on a product of two functions like the Leibnitz rule, except for a crucial sign difference. For example

$$\begin{aligned} D_x \, f \cdot g &= f_x g - f g_x, \\ D_x D_t \, f \cdot g &= f g_{xt} - f_x g_t - f_t g_x + f g_{xt}. \end{aligned}$$

Using the D-operator we can write (7) in the following condensed form

$$(D_x^4 + D_x D_t) \, F \cdot F = 0. \tag{9}$$

To summarize: what we needed in order to obtain the bilinear form (9) for (1) is a dependent variable transformation

$$u = 2\partial_x^2 \log F, \tag{10}$$

and we also had to integrate the equation once.

For a further discussion of bilinearization, see e.g., [6, 7]. Unfortunately the process of bilinearization is far from being algorithmic. It is even difficult to find out beforehand how many new independent and/or dependent variables are needed for the bilinearization. In fact for some equations the natural form may not be bilinear but perhaps trilinear [8]. Recently there have been some indications that singularity analysis can be used to reduce the guesswork: the number of dependent variables seems to be related to the number of singular manifolds [9].

One important property of equations in Hirota's bilinear form is their gauge invariance. One can show [8] that for a quadratic expression homogeneous in the derivatives, i.e., of the form $\sum_{i=0}^n c_i \, (\partial_x^i \, f) \, (\partial_x^{n-i} \, g)$, the requirement of gauge invariance under $f \to e^{kx} f$, $g \to e^{kx} g$ implies that the expression can be written in terms of Hirota derivatives. This gauge invariance can be taken as a starting point for further generalizations [8, 10].

Finally in this section we would like to list some useful properties of the bilinear derivative [5]. For P a polynomial,

$$P(D) f \cdot g \;=\; P(-D) g \cdot f, \tag{11}$$

$$P(D)1\cdot f = P(-\partial)f, \qquad P(D)f\cdot 1 = P(\partial)f, \tag{12}$$

$$P(D)e^{px}\cdot e^{qx} = P(p-q)e^{(p+q)x}, \tag{13}$$

$$\partial_x^2 \log f = (D_x^2 f\cdot f)/(2f^2), \tag{14}$$

$$\partial_x^4 \log f = (D_x^4 f\cdot f)/(2f^2) - 3(D_x^2 f\cdot f)^2/(2f^4). \tag{15}$$

3 Constructing multisoliton solutions

In this section we will construct soliton solutions for a class of equations. Properly speaking, the solutions should be called "solitary waves" until we can prove that they scatter elastically. However, since we are working towards integrable soliton equations, the solitary wave solutions will at some point become true solitons, so to make things simple we just call them solitons all the time.

3.1 The vacuum, and the one-soliton solution

Now that we have the KdV equation in the bilinear form, let us start constructing soliton solutions for it. In fact, it is equally easy to consider a whole class of bilinear equations of the form

$$P(D_x, D_y, ...)F\cdot F = 0, \tag{16}$$

where P is a polynomial in the Hirota partial derivatives D. We may assume that P is even, because the odd terms cancel due to the antisymmetry of the D-operator.

Let us start with the zero-soliton solution or the vacuum. We know that the KdV equation has a solution $u \equiv 0$ and now we want to find the corresponding F. From (10) we see that $F = e^{2\phi(t)x+\beta(t)}$ yields a u that solves (1), and in view of the gauge freedom we can choose $F = 1$ as our vacuum solution. It solves (16) provided that

$$P(0, 0, \ldots) = 0. \tag{17}$$

This is then the first condition that we have to impose on the polynomial P in (16).

The multisoliton solutions are obtained by finite perturbation expansions around the vacuum $F = 1$:

$$F = 1 + \epsilon\, f_1 + \epsilon^2\, f_2 + \epsilon^3\, f_3 + \cdots \tag{18}$$

Here ϵ is a formal expansion parameter. For the one-soliton solution (1SS) only one term is needed. If we substitute

$$F = 1 + \epsilon\, f_1 \tag{19}$$

into (16) we obtain

$$P(D_x, \ldots)\{1\cdot 1 + \epsilon\, 1\cdot f_1 + \epsilon\, f_1\cdot 1 + \epsilon^2\, f_1\cdot f_1\} = 0.$$

The term of order ϵ^0 vanishes because of (17). For the terms of order ϵ^1 we use property (12) so that, since now P is even, we get

$$P(\partial_x, \partial_y, \ldots) f_1 = 0. \tag{20}$$

The soliton solutions correspond to the exponential solutions of (20). For a 1SS we take an f_1 with just one exponential

$$f_1 = e^{\eta}, \quad \eta = px + qy + \ldots + \text{ const}, \tag{21}$$

and then (20) becomes the *dispersion relation* on the parameters $p, q, \ldots$

$$P(p, q, \ldots) = 0. \tag{22}$$

Finally, the order ϵ^2 term vanishes because

$$P(\vec{D}) e^{\eta} \cdot e^{\eta} = e^{2\eta} P(\vec{p} - \vec{p}) = 0,$$

by (17).

In summary, the 1SS is given by (19,21) where the parameters are constrained by (22). For KdV the dispersion relation is $q^3 = p$.

3.2 The two-soliton solution

The 2SS is built from two 1SS's, and one important principle is that for integrable systems one must be able to combine *any* pair of 1SS's built on top of the same vacuum. Thus if we have two 1SS's, $F_1 = 1 + e^{\eta_1}$ and $F_2 = 1 + e^{\eta_2}$, we should be able to combine them into a form $F = 1 + f_1 + f_2$, where $f_1 = e^{\eta_1} + e^{\eta_1}$. Gauge invariance suggest that we should try the combination

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{23}$$

where there is just one arbitrary constant A_{12}. Substituting this into (16) yields

$$\begin{array}{rcccccccccl}
P(D)\{ & 1 \cdot 1 & + & 1 \cdot e^{\eta_1} & + & 1 \cdot e^{\eta_2} & + & \underline{A_{12}\, 1 \cdot e^{\eta_1+\eta_2}} & + & \\
& e^{\eta_1} \cdot 1 & + & e^{\eta_1} \cdot e^{\eta_1} & + & \underline{e^{\eta_1} \cdot e^{\eta_2}} & + & A_{12}\, e^{\eta_1} \cdot e^{\eta_1+\eta_2} & + & \\
& e^{\eta_2} \cdot 1 & + & \underline{e^{\eta_2} \cdot e^{\eta_1}} & + & e^{\eta_2} \cdot e^{\eta_2} & + & A_{12}\, e^{\eta_2} \cdot e^{\eta_1+\eta_2} & + & \\
& \underline{A_{12} e^{\eta_1+\eta_2} \cdot 1} & + & A_{12} e^{\eta_1+\eta_2} \cdot e^{\eta_1} & + & A_{12} e^{\eta_1+\eta_2} \cdot e^{\eta_2} & + & A_{12}^2 e^{\eta_1+\eta_2} \cdot e^{\eta_1+\eta_2} & & \} = 0.
\end{array}$$

In this equation all non-underlined terms vanish due to (17,22). Since P is even, the underlined terms combine as $2A_{12} P(\vec{p}_1 + \vec{p}_2) + 2P(\vec{p}_1 - \vec{p}_2) = 0$, from which A_{12} can be solved as

$$A_{12} = -\frac{P(\vec{p}_1 - \vec{p}_2)}{P(\vec{p}_1 + \vec{p}_2)}. \tag{24}$$

The important thing about this result is that we were able to construct a two-soliton solution for a huge class of equations, namely all those whose bilinear form is of type (16). In particular this includes many non-integrable systems.

3.3 Multi-soliton solutions

The above shows that for the KdV class (16) the existence of 2SS is not strongly related to integrability, but it turns out that the existence on 3SS is very restrictive.

A 3SS should start with $f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$ and, if the above is any guide, contain terms up to f_3. If we now use the requirement that the solution should reduce to a 2SS when the third soliton goes to infinity (which corresponds to $\eta_k \to \pm\infty$) then one finds that F must have the form

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ &\qquad + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}. \end{aligned} \tag{25}$$

Note in particular that this expression contains no additional freedom. The parameters p_i are only required to satisfy the dispersion relation (22) and the phase factors A were already determined (24). This extends to NSS [11]:

$$F = \sum_{\substack{\mu_i=0,1 \\ 1\le i\le N}} \exp\left(\sum_{1\le i<j\le N} \varphi(i,j)\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\eta_i \right), \tag{26}$$

where ($A_{ij} = e^{\varphi(i,j)}$). Thus the ansatz for a NSS is completely fixed and the requirement that it be a solution of (16) implies conditions on the equation itself. Only for integrable equations can we combine solitons in this simple way. More precisely, let us make the

Definition: *A set of equations written in the Hirota bilinear form is* **Hirota integrable**, *if one can combine any number N of one-soliton solutions into an NSS, and the combination is a finite polynomial in the e^{η}'s involved.*

In all cases known so far, Hirota integrability has turned out to be equivalent to more conventional definitions of integrability.

4 Searching for integrable evolution equations.

Since the existence of a 3SS is very restrictive, one can use it as a method for searching for new integrable equations. All search methods depend on some initial assumptions about the structure. In this case we assume that the nonlinear PDE can be put into a bilinear form of type (16), but no assumptions are made for example on the number of independent variables.

4.1 KdV

If one now substitutes (25) into (16) one obtains the condition

$$\sum_{\sigma_i=\pm 1} P(\sigma_1\vec{p}_1+\sigma_2\vec{p}_2+\sigma_3\vec{p}_3) \times P(\sigma_1\vec{p}_1-\sigma_2\vec{p}_2)P(\sigma_2\vec{p}_2-\sigma_3\vec{p}_3)P(\sigma_3\vec{p}_3-\sigma_1\vec{p}_1) \doteq 0, \tag{27}$$

where the symbol $\doteq 0$ means that the equation is required to hold only when the parameters $\vec{p}_i$ satisfy the dispersion relation $P(\vec{p}_i)=0$.

In order to find possible solutions of (27) we made a computer assisted study [12] and the result was that the only genuinely nonlinear equations that solved (27) were

$$(D_x^4-4D_xD_t+3D_y^2)F\cdot F = 0, \tag{28}$$

$$(D_x^3D_t+aD_x^2+D_tD_y)F\cdot F = 0, \tag{29}$$

$$(D_x^4-D_xD_t^3+aD_x^2+bD_xD_t+cD_t^2)F\cdot F = 0, \tag{30}$$

$$(D_x^6+5D_x^3D_t-5D_t^2+D_xDy)F\cdot F = 0. \tag{31}$$

and their reductions. These equations also have 4SS and they all pass the Painlevé test [14]. Among them we recognize the Kadomtsev-Petviashvili (containing KdV and Boussinesq) (28), Hirota-Satsuma-Ito (29) and Sawada-Kotera-Ramani (31) equations; they appear in the Jimbo-Miwa classification [13]. The only new equation is (30). It is somewhat mysterious. It has not been identified within the Jimbo-Miwa classification because it has no nontrivial scaling invariances, furthermore we do not know its Lax pair or Bäcklund transformation. But it does have at least 4SS, and it passes the Painlevé test [14].

4.2 mKdV and sG

As was mentioned before, Hirota's bilinear method has been applied to many other equations beside KdV. Here we would like to mention briefly some of them.

For example the modified Korteweg–de Vries (mKdV) and sine–Gordon (sG) equations have a bilinear form of the type

$$\begin{cases} B(D_{\vec{x}})\, G\cdot F = 0, \\ A(D_{\vec{x}})(F\cdot F+G\cdot G) = 0, \end{cases} \tag{32}$$

where A is even and B either odd (mKdV) or even (sG). For mKdV we have $B = D_x^3+D_t$, $A=D_x^2$, and for SG, $B=D_xD_y-1$, $A=D_xD_y$. This class of equations also has 2SS for any choice of A and B. If B is odd one can make a rotation $F=f+g$, $G=i(f-g)$ after which the pair (32) becomes $B\,g\cdot f=0$, $A\,g\cdot f=0$.

In principle the pair (32) can have two different kinds of solitons,

$$\begin{cases} F = 1 + e^{\eta_A},\ G = 0, & \text{with dispersion relation } A(\vec{p}) = 0, \\ F = 1,\ G = e^{\eta_B}, & \text{with dispersion relation } B(\vec{p}) = 0. \end{cases} \tag{33}$$

In mKdV and SG the A polynomial is too trivial to make the first kind of soliton interesting. In [15, 7] we searched for polynomials A and B for which any set of three solitons could be combined for a 3SS. The final result contains 5 equation, of mKdV type, three of them have a nonlinear B polynomial but a factorizable A part (and hence only one kind of soliton with B acting as the dispersion relation),

$$\left\{ \begin{array}{rcl} (aD_x^7 + bD_x^5 + D_x^2 D_t + D_y)\, G \cdot F &=& 0, \\ D_x^2\, G \cdot F &=& 0, \end{array} \right. \tag{34}$$

$$\left\{ \begin{array}{rcl} (aD_x^3 + bD_x^3 + D_y)\, G \cdot F &=& 0, \\ D_x D_t\, G \cdot F &=& 0, \end{array} \right. \tag{35}$$

$$\left\{ \begin{array}{rcl} (D_x D_y D_t + aD_x + bD_t)\, G \cdot F &=& 0, \\ D_x D_t\, G \cdot F &=& 0. \end{array} \right. \tag{36}$$

For a discussion of the nonlinear versions of the last two equations, see [16].

In two cases both A and B are nonlinear enough to support solitons, note that the B polynomials are the same and that the A parts have already appeared in the KdV list:

$$\left\{ \begin{array}{rcl} (D_x^3 + D_y)\, G \cdot F &=& 0, \\ (D_x^3 D_t + aD_x^2 + D_t D_y)\, G \cdot F &=& 0, \end{array} \right. \tag{37}$$

$$\left\{ \begin{array}{rcl} (D_x^3 + D_y)\, G \cdot F &=& 0, \\ (D_x^6 + 5D_x^3 D_y - 5D_y^2 + D_t D_x)\, G \cdot F &=& 0. \end{array} \right. \tag{38}$$

Two equations of sine-Gordon type were also found:

$$\left\{ \begin{array}{rcl} (D_x D_t + b)\, G \cdot F &=& 0, \\ (D_x^3 D_t + 3bD_x^2 + D_t D_y)(F \cdot F + G \cdot G) &=& 0, \end{array} \right. \tag{39}$$

$$\left\{ \begin{array}{rcl} (aD_x^3 D_t + D_t D_y + b)\, G \cdot F &=& 0, \\ D_x D_t (F \cdot F + G \cdot G) &=& 0. \end{array} \right. \tag{40}$$

4.3 nlS

A similar search was performed [17, 7, 18] on equations of nonlinear Schrödinger (nlS) type,

$$\left\{ \begin{array}{rcl} B(D_{\vec{x}})\, G \cdot F &=& 0, \\ A(D_{\vec{x}})\, F \cdot F &=& |G|^2, \end{array} \right. \tag{41}$$

where F is real and G complex. Again two kinds of solitons exist,

$$\begin{cases} F = 1 + e^{\eta_A},\ G = 0, & \text{with dispersion relation } A(\vec{p}) = 0, \\ F = 1 + Ke^{\eta_B + \eta_B^*},\ G = e^{\eta_B}, & \text{with dispersion relation } B(\vec{p}) = 0. \end{cases} \tag{42}$$

In this case the existence of a 2SS is not automatic because the 1SS already involves terms of order ϵ^2 and a 2SS therefore ϵ^4, whereas in the previous cases ϵ^2 contributions were sufficient for 2SS.

Three equations were found in this search:

$$\begin{cases} (D_x^2 + iD_y + c)\, G \cdot F &= 0, \\ (a(D_x^4 - 3D_y^2) + D_x D_t)\, F \cdot F &= |G|^2, \end{cases} \tag{43}$$

$$\begin{cases} (i\alpha D_x^3 + 3cD_x^2 + i(bD_x - 2dD_t) + g)\, G \cdot F &= 0, \\ (\alpha D_x^3 D_t + aD_x^2 + (b + 3c^2) D_x D_t + dD_t^2)\, F \cdot F &= |G|^2, \end{cases} \tag{44}$$

$$\begin{cases} (i\alpha D_x^3 + 3D_x D_y - 2iD_t + c)\, G \cdot F &= 0, \\ (a(\alpha^2 D_x^4 - 3D_y^2 + 4\alpha D_x D_t) + bD_x^2)\, F \cdot F &= |G|^2. \end{cases} \tag{45}$$

Perhaps the most interesting new equation above is the combination in (45) of the two most important $(2+1)$-dimensional equations, Davey-Stewartson and Kadomtsev-Petviashvili equations, see [18], page 315.

The above lists contains many equations of which nothing else is known other than that they have 3SS and 4SS. This alone suggests that they are good candidates for integrable $2+1$ dimensional equations, but connections to other definitions of integrability (like Lax pairs) are still open.

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