Math 20 Exam 3 (show all work) Name

1. (10 points) Find the particular solution to the problem: $\frac{dy}{dx}=\frac{y^2}{x^2}$, $y=1$ when $x=1$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x^2}$$

$$-\frac{1}{y} = -\frac{1}{x} + c.$$

$$-1 = -1 + c \Rightarrow c = 0$$

$$\Rightarrow -\frac{1}{y} = -\frac{1}{x} \Rightarrow y = x$$

2. (15 points) Find the general solution for the differential equation: $y - x\frac{dy}{dx} = x^3$

$$\frac{dy}{dx} - \frac{y}{x} = -x^2$$

$$y(x) = e^{\int \frac{1}{x}dx} \int e^{-\int \frac{1}{x}dx}(-x^2)\,dx = e^{\ln x}\int e^{-\ln x}\cdot(-x^2)\,dx$$

$$= -x\int \frac{1}{x}\cdot x^2\,dx = -x\int x\,dx = -x\left[\frac{1}{2}x^2 + c\right] = -\frac{1}{2}x^3 - cx$$

3. (15 points) Mike deposits \$10,000 in an IRA at 10% interest compounded continuously for his retirement in 20 years. He intends to make continuous deposits at the rate of \$3000 a year until he retires. How much will he have accumulated at that time?

$y(t)$: account balance after t years

$$\frac{dy}{dt} = 0.1y + 3000 \Rightarrow y(t) = e^{\int 0.1dt}\int e^{-\int 0.1dt}\cdot 3000\,dt$$

$$= e^{0.1t}\int 3000e^{-0.1t}\,dt$$

$$= e^{0.1t}\left[-30000e^{-0.1t} + c\right]$$

$$= ce^{0.1t} - 30000.$$

$$y(0) = c - 30000 = 10000$$

$$\Rightarrow c = 40000$$

$$\Rightarrow y(t) = 40000e^{0.1t} - 30000$$

$$y(20) = 40000e^2 - 30000 = 265{,}56$$

4. (15 points) For the function

$$f(x) = e^{-x}, \qquad 0 \le x < \infty$$

(1) determine whether it is a probability density function or not.

(2) if yes, determine the probability $P(3 \le x \le 5)$

(1) $f(x) \ge 0$. $\int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$

$\Rightarrow f(x)$ is a probability density function

(2) $P(3 \le x \le 5) = \int_3^5 e^{-x}\,dx = -e^{-x}\Big|_3^5$

$$= -e^{-5} + e^{-3} = 0.043$$

5. (15 points) For the probability density function

$$f(x)=3x^{-4}, \quad 1 \le x < \infty,$$

determine its expected value, the variance and standard deviation.

$$\mu = \int_1^{\infty} x f(x)\,dx = \int_1^{\infty} x \cdot 3x^{-4}\,dx = 3\int_1^{\infty} x^{-3}\,dx = 3\cdot\left(-\tfrac{1}{2}\right)x^{-2}\Big|_1^{\infty} = \frac{3}{2}$$

$$Var(x) = \int_1^{\infty} x^2 f(x)\,dx - \mu^2 = \int_1^{\infty} x^2\cdot 3x^{-4}\,dx - \mu^2 = 3\int_1^{\infty} x^{-2}\,dx - \mu^2 = 3(-x^{-1})\Big|_1^{\infty} - \mu^2$$

$$= 3 - \left(\tfrac{3}{2}\right)^2 = 0.75$$

$$\sigma = \sqrt{0.75} = 0.866$$

6. (15 points) For the probability density function

$$f(x)=0.1, \quad 0 \le x < 10,$$

determine its mean, standard deviation, and the probability that x is between the mean and one standard deviation above the mean.

$$\mu = \int_0^{10} x f(x)\,dx = \int_0^{10} 0.1x\,dx = \frac{0.1}{2}x^2\Big|_0^{10} = 5$$

$$Var(x) = \int_0^{10} x^2 f(x)\,dx - \mu^2 = \int_0^{10} 0.1x^2\,dx - \mu^2 = \frac{0.1}{3}x^3\Big|_0^{10} - \mu^2 = \frac{100}{3} - 5^2$$

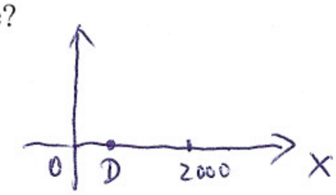
$$= 8.3333$$

$$\sigma = \sqrt{8.3333} = 2.887$$

$$P(5<x<5+2.887) = P(5<x<7.887) = 0.1\times(7.887-5) = 0.2887$$

7. (15 points) An insurance policy is written to cover a loss X, where X is uniformly distributed between \$0 and \$2000. At what level must a deductible be set in order for the expected payment to be 30% of what it would be with no deductible?

$$f(x) = \frac{1}{2000}, \qquad 0 \le x \le 2000$$



No deductible. $\Rightarrow \quad \mu = 1000$

With deductible D,

$$\mu = \int_D^{2000} (x-D)\cdot f(x)\,dx = \int_D^{2000} \frac{1}{2000}(x-D)\,dx = \frac{1}{2000}\int_D^{2000} \left(\tfrac{1}{2}x^2 - Dx\right) dx$$

$$= \frac{1}{2000}\left[\frac{1}{2}\cdot 2000^2 - 2000D - \left(\frac{1}{2}D^2 - D^2\right)\right] = \frac{1}{2000}\left[2000000 - 2000D + \frac{1}{2}D^2\right]$$

$$= 0.3 \cdot 1000 = 300$$

$$\Rightarrow \quad \frac{1}{2000}\left[2000000 - 2000D + \frac{1}{2}D^2\right] = 300 \Rightarrow \frac{1}{2}D^2 - 2000D + 2000000 = 600000$$

$$\Rightarrow \quad D = 904.6$$