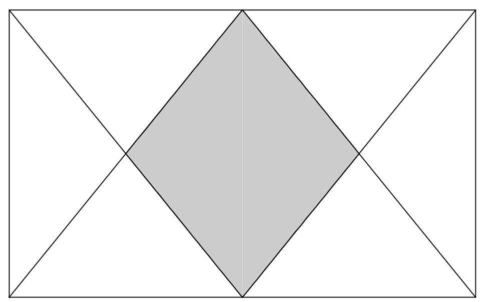
UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FIFTIETH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 13, 2007

1) Given real numbers x, y and z, define $\boxed{x,\ y,\ z} = \dfrac{x\,y + y\,z + z\,x}{x^2 + y^2 + z^2}$. Evaluate $\boxed{3,\ 2,\ -4}$.

2) Simplify $\dfrac{\sqrt{32} - \sqrt{2}}{\sqrt{32} + \sqrt{2}}$.

3) The midpoints of the longer sides of a 2 by 4 rectangle are joined to the opposite vertices as indicated in the figure. Find the area of the shaded quadrilateral.



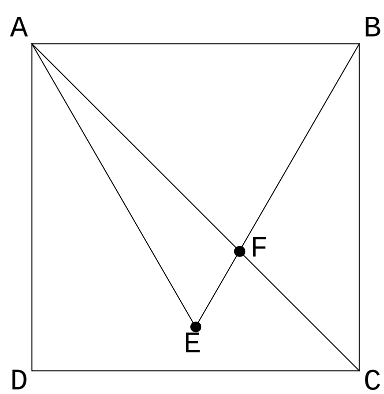
4) Given three consecutive integers, the difference between the cubes of the two largest integers is 666 more than the difference between the cubes of the two smallest integers. What is the largest integer ?

5) If w is 10% larger than x, x is 20% larger than y and y is 25% smaller than z, by what percent is w smaller than z ?

6) A ladder leans against a wall. The top of the ladder is 8 feet above the ground. If the bottom of the ladder is then moved 2 feet farther from the wall, the top of the ladder will rest against the foot of the wall. How long is the ladder ?

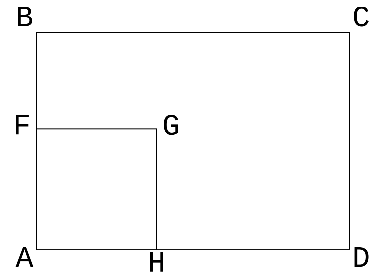
7) ABCD is a square, ABE is an equilateral triangle and F is the point of intersection of $\overline{AC}$ and $\overline{BE}$.

Find the degree measure of $\angle$ EAF.



8) Joe has a bag of marbles. Joe gives Larry half of his marbles and two more. Joe then gives Doug half of the marbles he has left and two more. Finally, Joe gives Jack half of the marbles he has left and two more. Joe has one marble remaining in his bag. With how many marbles did Joe start ?

9) The square AFGH is cut out of the rectangle ABCD, leaving an area of 92 in^2. If FB = 4 inches and DH = 8 inches, find the original area of ABCD.



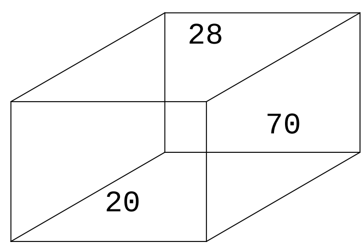
10) Suppose that 60 percent of the population has a particular virus. A medical test accurately detects the virus in 90 percent of the cases in which the patient has the virus, but falsely detects the virus in 20 percent of the cases in which the patient does not have the virus. If the same patient is tested twice, the results are said to be inconsistent if the results of the two tests do not agree. If the test is administered twice to all patients, how many patients out of 250 would expect to get inconsistent results ?

11) Find all real solutions of $9^x + 2\,(3^{x+2}) = 243$.

12) Suppose that $\sin(2x) = \frac{1}{\sqrt{7}}$. Express $\sin^4(x) + \cos^4(x)$ as a rational number in lowest terms.

13) For each positive integer k, let a_k be the sum of the first k positive integers. If exactly x of the a_k consist of one digit, exactly y of the a_k consist of two digits and exactly z of the a_k consist of three digits, what is the product $x \cdot y \cdot z$?

14) A rectangular solid has a top face with surface area of 28 square feet, a front face with surface area of 20 square feet and a side face with surface area of 70 square feet. What is the volume of this solid ?



15) John rolls two standard six-sided dice. Given that the sum of the numbers on the top faces is not 6, what is the probability that the sum is 7 ? Express your answer as a rational number in lowest terms.

16) A hardware store sells three different kinds of widgets. Green widgets cost 25¢ each, blue widgets cost \$1 each and red widgets are \$3 each. Traci bought 75 widgets for a total cost of exactly \$75. What is the greatest number of green widgets she could have bought?

17) Find the area of the convex polygon in the plane with vertices at the points whose coordinates are $(-2, 3)$, $(1, 10)$, $(5, 10)$, $(8, 7)$ and $(4, 0)$.

18) Let θ be an acute angle such that $\tan(2\theta) + \cot(2\theta) = 10$. Express $\sin(4\theta)$ as a rational number in lowest terms.

19) A box contains 6 red balls, 4 blue balls and 2 green balls. Three balls are randomly drawn (without replacement) from the box. What is the probability that the three balls selected are all the same color? Express your answer as a rational number in lowest terms.

20) Five members of a basketball team are weighed. An average weight is calculated after each player is weighed. If the average weight increases by 2 pounds each time, how much heavier is the last player than the first ?

21) The lines $x - 2y = 2$, $-3x + y = 4$ and $2x + y = 4$ intersect in pairs to determine the vertices of a triangle. Find the area of this triangle.

22) Rectangle A has vertices at $(0, 0)$, $(0, 6)$, $(-4, 6)$ and $(-4, 0)$. Rectangle B has vertices at $(2, 0)$, $(20, 0)$, $(20, 8)$ and $(2, 8)$. A line divides each rectangle into two regions of equal area. What is the slope of this line ?

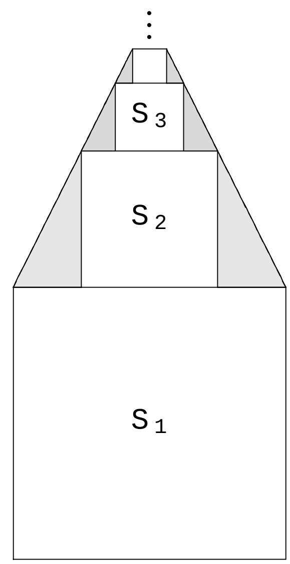
23) Suppose that P is a point that lies outside of two concentric circles C_1 and C_2, with C_1 inside C_2. Suppose that T is a point on C_1 such that $\overline{PT}$ is tangent to C_1 and PT = 17; suppose that S is a point on C_2 so that $\overline{PS}$ is tangent to C_2 and PS = 15. If Q is the point on C_2 that is on $\overline{PT}$ (with Q between P and T), find TQ.

24) Find the sum of all real numbers x such that $|x - 2006| + |x - 2007| = 3$.

25) Suppose that a, b and c are positive integers such that $a \log_{144}(3) + b \log_{144}(2) = c$. What is the value of $\frac{a+b}{c}$?

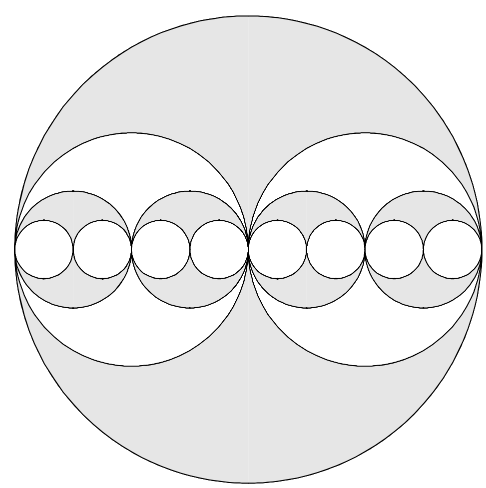
26) Let $f(x) = ax + b$. Find all real values of a and b such that $f(f(f(1))) = 29$ and $f(f(f(0))) = 2$.

27) Let S_1 be a square of side length 1. Let S_2 be a square of side length $\frac{1}{2}$ centered on the top side of S_1. Construct triangles by joining the top vertices of squares S_1 and S_2. Let S_3 be a square of side length $\frac{1}{4}$ centered on the top side of S_2. Construct triangles by joining the top vertices of squares S_2 and S_3. If this construction is continued without end, find the sum of the areas of all of the triangles.



28) If $N^{\log_5(7)} = 25$, find $N^{(\log_5(7))^2}$.

29) Let C_1 be a circle of radius 1, let C_2 be a pair of congruent circles inscribed in C_1, let C_4 be four congruent circles inscribed in circles C_2 and let C_8 be eight congruent circles inscribed in circles C_4. The centers of all circles are on a diameter of C_1. Find the area inside C_1 and outside C_2 plus the area inside C_4 and outside C_8.



30) Let S be the set of all five–digit positive integers formed from the digits in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ without repetition. Find the number of these integers in which the digits 3, 5 and 7 appear in order (not necessarily consecutively) from left to right. For example, 93157 is such an integer.

31) Alice, Bob, Carl and Dave decide to play a game in which they take turns spinning a wheel on which the numbers 1, 2, 3 and 4 are equally likely to be selected. The winner is the first person who spins a 3. The order of play is Alice, Bob, Carl and Dave, repeated until a winner is decided. Find the probability that Alice wins. Express your answer as a rational number in lowest terms.

32) In triangle ABC, AC = 12. If one of the trisectors of angle B is the median to $\overline{AC}$ and the other trisector of angle B is the altitude to $\overline{AC}$, find the area of triangle ABC.

33) Given a finite sequence of n real numbers, let S_k be the sum of the first k terms of the sequence. The Cesaro sum of this sequence is defined to be $\frac{S_1 + S_2 + \cdots + S_n}{n}$. Suppose that the Cesaro sum of $s = \{ a_1, a_2, \cdots a_{49} \}$ is 100. Find the value of a_0 so that the Cesaro sum of $s^* = \{ a_0, a_1, a_2, \cdots a_{49} \}$ is 101.

34) In a rectangular solid, let d be the distance from one vertex to the diagonally opposite vertex. If the total edge length is 124 and the total surface area is 622, find d^2.

35) Find the largest positive integer p such that 5^7 can be expressed as the sum of p consecutive positive integers.

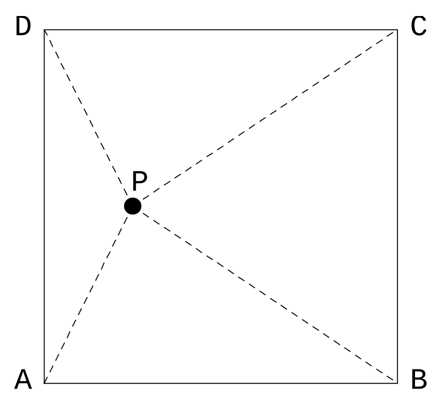
36) Find the number of ordered pairs (a, b) of positive integers such that $\log_a(b) + 10 \log_b(a) = 7$ and $2 \le a \le b \le 2007$.

37) If $f(x) = x^3 - 4x^2 + 18x - 9$, $f(a) = 0$, $f(b) = 0$, $f(c) = 0$ and a, b and c are all distinct, find $f(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$.

38) Find bases a and b such that $386_a = 272_b$ and $146_a = 102_b$.

39) Let $p(x) = x^3 + x^{12} + x^{21} + x^{30} + \cdots + x^{2010} = \sum_{k=0}^{223} x^{3+9k}$ and $q(x) = x^6 - x^3$.
Find the remainder when $p(x)$ is divided by $q(x)$.

40) Let ABCD be a square with A= (0 , 0) , B = (5 , 0) , C = (5 , 5) and D = (0 , 5). Let P be a point on the line $y = 2x$. Find the value of x such that the sum of the squares of the distances from P to the vertices of the square is a minimum.



41) In Δ ABC, AB = 7, BC = 5 and CA = 6. Locate points P_1, P_2, P_3 and P_4 on BC so that this side is partitioned into five congruent segments, each of length 1. For k = 1, 2, 3 and 4 , let $q_k = AP_k$. Find $q_1^2 + q_2^2 + q_3^2 + q_4^2$.