UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
FIFTY-THIRD ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 10, 2010

1) Express $\dfrac{\frac{1}{2}-\frac{3}{4}}{\frac{5}{6}-\frac{7}{8}}$ as a rational number in lowest terms.

$$\frac{\frac{1}{2}-\frac{3}{4}}{\frac{5}{6}-\frac{7}{8}}\;\frac{24}{24}=\frac{12-18}{20-21}=\mathbf{6}$$

2) Simplify the expression $\left(2^{-2}+3^{-2}+12^{-1}\right)^{3/2}$. Express your answer as a rational number in lowest terms.

$$\left(2^{-2}+3^{-2}+12^{-1}\right)^{3/2}=\left[\frac{1}{4}+\frac{1}{9}+\frac{1}{12}\right]^{3/2}=\left[\frac{9+4+3}{36}\right]^{3/2}=\left(\frac{16}{36}\right)^{3/2}=\left(\frac{4}{9}\right)^{3/2}=\mathbf{\frac{8}{27}}$$

3) Express $\dfrac{1072^2-938^2}{2010^2}$ as a rational number in lowest terms.

$$\frac{1072^2-938^2}{2010^2}=\frac{(1072+938)(1072-938)}{2010^2}=\frac{2010\,(134)}{2010^2}=\frac{134}{2010}=\frac{2\cdot 67}{2\cdot 3\cdot 5\cdot 67}=\mathbf{\frac{1}{15}}$$

4) Larry has a sum of money that he would like to donate to charity. He wants to give \$25 to each of his favorite charities, but he will be short \$10. He decides to give only \$20 to each of his favorite charities, which will leave him with \$25. Find the sum of money Larry wishes to donate to charity.

Let x = amount and n = number of charities

$25n = x + 10$
$20n = x - 25$ subtract

$5n = 35 \implies n = 7$

$20(7) = x - 25 \implies$ **x = 165**

5) If $x + y = 4$ and $x\,y = -12$, find the value of $x^2 + 5xy + y^2$.

$(x+y)^2 = x^2 + 2xy + y^2 = 4^2 = 16$

$x^2 + 5xy + y^2 = x^2 + 2xy + y^2 + 3xy = 16 + 3(-12) = 16 - 36 =$ **– 20**

6) What is the smallest positive integer n such that 450n is a perfect cube?

$450n = 5^2\cdot 3^2\cdot 2\cdot n$

Thus $n = 5\cdot 3\cdot 2^2 = 15\cdot 4 =$ **60**

7) Find the largest integer n such that $\dfrac{n^2-38}{n+1}$ is an integer.

$$\frac{n^2-38}{n+1}=\frac{n^2+2n+1-(2n+1)-38}{n+1}=\frac{(n+1)^2-2n-1-1+1-38}{n+1}=\frac{(n+1)^2-2(n+1)-37}{n+1}$$

$$= n+1-2-\frac{37}{n+1}$$

To be an integer (n+1) must divide 37 $\implies n + 1 = 37 \implies$ **n = 36**

8) Define the function $T(n) = \begin{cases} \frac{3n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$. For example, T(13) = 20 and T(8) = 4.

What is the smallest value of k such that $T^k(113) = \underbrace{(T \circ T \circ \cdot \cdot \cdot \circ T)}_{k - \text{iterations}}(113) = 1$?

$T(113) = \frac{3(113)+1}{2} = 170$

$T^2(113) = T(170) = \frac{170}{2} = 85$

$T^3(113) = T(85) = \frac{3(85)+1}{2} = 128 = 2^7$

7 more steps gives a value of 1 $\Longrightarrow$ **k = 10**

9) Find the value of $\sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}}$.

$x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}}$

$x^2 = 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}} = 3 + x$

$x^2 - x - 3 = 0 \Longrightarrow x = \frac{1 \pm \sqrt{1-4(-3)}}{2} = \frac{1 \pm \sqrt{13}}{2}$

$x \geq 0 \Longrightarrow$ $\mathbf{x = \frac{1+\sqrt{13}}{2}}$

10) Mr. Kost has 20 students in his morning class and 30 students in his afternoon class. He gave the same test to both classes. The average grade on the test for his morning class was 70%. The average grade for the two classes combined was 79%. What was the average grade for his afternoon class?

Morning ave = 70 Afternoon ave = v

$\frac{20(70) + 30v}{50} = 79$

$20(70) + 30v = 50(79) \Longrightarrow 30v = 50(79) - 20(70) = 3950 - 1400 = 2550$

$v = \frac{2550}{30} =$ **85**

11) Find all real values of x such that $x \ln\left(e^{2x+7}\right) = 30$.

$x(2x + 7) = 30 \Longrightarrow 2x^2 + 7x - 30 = 0$

$(2x - 5)(x + 6) = 0 \Longrightarrow$ $\mathbf{x = \frac{5}{2}, -6}$

12) Define the operation $\otimes$ by $a \otimes b = \frac{a}{b} + a\,b$. Find the value of a such that $2 \otimes 3 = a \otimes 2$.

$2 \otimes 3 = a \otimes 2$

$\frac{2}{3} + 6 = \frac{a}{2} + 2a$

$4 + 36 = 3a + 12a \implies 15a = 40 \implies a = \frac{40}{15} = \mathbf{\frac{8}{3}}$

13) Find the positive integer base b such that $(16_b)^2 = 331_b$.

$(b + 6)^2 = 3b^2 + 3b + 1 \implies b^2 + 12b + 36 = 3b^2 + 3b + 1$

$2b^2 - 9b - 35 = 0 \implies (2b + 5)(b - 7) = 0 \implies b = -\frac{5}{2}, 7$

b = 7

14) Find the exact value of $\cos^3(15°)\sin(15°) - \cos(15°)\sin^3(15°)$.

$\cos^3(15°)\sin(15°) - \cos(15°)\sin^3(15°) = \sin(15°)\cos(15°)(\cos^2(15°) - \sin^3(15°))$

$\cos^3(15°)\sin(15°) - \cos(15°)\sin^3(15°) = \frac{1}{2}\sin(30°)\cos(30°) = \frac{1}{2}\frac{1}{2}\frac{\sqrt{3}}{2} = \mathbf{\frac{\sqrt{3}}{8}}$

15) Find all ordered pairs (x,y) that satisfy the system of equations $\begin{cases} \frac{4}{x} + \frac{5}{y^2} = 12 \\ \frac{3}{x} + \frac{7}{y^2} = 22 \end{cases}$.

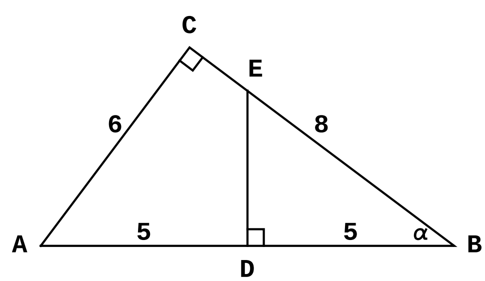
$\begin{cases} \frac{12}{x} + \frac{15}{y^2} = 36 \\ \frac{-12}{x} - \frac{28}{y^2} = -88 \end{cases}$ add

$-\frac{13}{y^2} = -52 \implies y^2 = \frac{1}{4} \implies y = \pm\frac{1}{2}$

Sub is first equation $\frac{4}{x} + \frac{5}{\frac{1}{4}} = 12 \implies \frac{4}{x} + 20 = 13 \implies \frac{1}{x} = 3 - 5 = -2 \implies x = -\frac{1}{2}$

$(x,y) = \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$

16) If the figure, AC = 6, BC = 8, AB = 10, D is the midpoint of $\overline{AB}$ and $\overline{DE} \perp \overline{AB}$. Find the length DE.



$\tan(\alpha) = \frac{6}{8} = \frac{DE}{5} \implies DE = \frac{30}{8} \implies$ **$DE = \frac{15}{4}$**

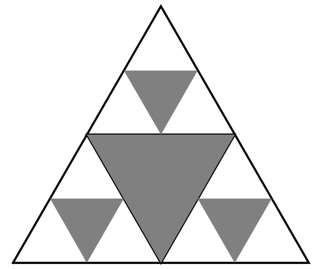
17) What is the coefficient of $a^3 b^2 c^3$ in the expansion of $(a + b + c)^8$?

$(a + b + c)^8 = \sum_{k=0}^{8} \binom{8}{k} a^k (b + c)^{8-k}$ Want k = 3

$\binom{8}{3} a^3 (b + c)^5 = \binom{8}{3} a^3 \sum_{j=0}^{5} \binom{5}{j} b^j c^{5-j}$ Want j = 2

$\binom{8}{3}\binom{5}{2} = \frac{8!}{3!\cdot 5!}\frac{5!}{2!\cdot 3!} = \frac{8!}{3!\cdot 2!\cdot 3!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3!}{3!\cdot 2!\cdot 3!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{6\cdot 2} = 8\cdot 7\cdot 5\cdot 2 =$ **560**

18) An equilateral triangle is hollowed out by connecting the midpoints of each side and removing the central triangle. The process is repeated on each of the three new smaller equilateral triangles. The shaded triangles represent the total area removed as a result of the first two hollowing out actions. This removal process is repeated twice more. If a random point is chosen from the original triangle, what is the probability that it will be removed during the four hollowing out actions? Express your answer as a rational number in lowest terms.



Let A = area of the triangle.

Step 1 Remove $\frac{1}{4}$A

Step 2 Remove $\frac{1}{4}A + 3\left(\frac{1}{16}\right)A$

Step 3 Remove $\frac{1}{4}A + 3\left(\frac{1}{16}\right)A + 9\left(\frac{1}{64}\right)A$

Step 4 Remove $\frac{1}{4}A + 3\left(\frac{1}{16}\right)A + 9\left(\frac{1}{64}\right)A + 27\left(\frac{1}{256}\right)A = \left(\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256}\right)A$

$$= \left(\frac{64+48+36+27}{256}\right)A = \frac{175}{256}A$$

$p = \frac{\frac{175}{256}A}{A} = \mathbf{\frac{175}{256}}$

19) The perimeter of an equilateral triangle is increased by 6 cm to form a new equilateral triangle. If the area of the new equilateral triangle is three times the area of the original equilateral triangle, find the length of a side of the new equilateral triangle.

Area of equilateral triangle of side s = $\frac{\sqrt{3}}{4}s^2$

$\frac{\sqrt{3}}{2}(s+2)^2 = 3 \cdot \frac{\sqrt{3}}{4}s^2$

$s^2 + 2s + 4 = 3s^2$

$2s^2 - 4s - 4 = 0$

$s^2 - 2s - 2 = 0 \implies s = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$

$s = 1 + \sqrt{3} \implies s + 2 = \mathbf{3 + \sqrt{3}}$

20) Find all real roots of the equation $(x-2)^4 - 25x(x-4) + 44 = 0$.

Let y = x + 2

$(x-2)^4 - 25x(x-4) + 44 = 0 = y^4 - 25(y+2)(y-2) + 44$

$y^4 - 25(y^2-4) + 44 = 0$

$y^4 - 25y^2 + 144 = 0$

$(y^2-9)(y^2-16) = 0$

$y = \pm 3, \pm 4$

$x = y + 2 \implies \mathbf{x = 5, -1, 6, -2}$

21) Let p, q, r ≥ 2 be integers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. What is the smallest value of p + q + r?

Take $2 \le p \le q \le r$

If p = 2 the q ≥ 3, p = 2 and q = 3 $\Longrightarrow$ r ≥ 7 $\Longrightarrow$ p + q + r ≥ 12

If p = 2 the q ≥ 3, p = 2 and q = 4 $\Longrightarrow$ r ≥ 5 $\Longrightarrow$ p + q + r ≥ 11

If p = 3 the q ≥ 3, p = 3 and q = 3 $\Longrightarrow$ r ≥ 5 $\Longrightarrow$ p + q + r ≥ 12

If p ≥ 4 then p + q + r ≥ 12

Thus minimum value of p + q + r = **11**

22) How many ways can the letters in **NOODLES** be arranged so that the consonants are in alphabetical order (not necessarily consecutively)?

Place DLNS in $\binom{7}{4}$ ways. Place E in $\binom{3}{1}$ ways.

$$\binom{7}{4}\cdot\binom{3}{1} = \frac{7!}{4!\cdot 3!}\cdot 3 = \frac{7\cdot 6\cdot 5\cdot 4!\cdot 3}{4!\cdot 6} = 7\cdot 5\cdot 3 = \mathbf{105}$$

23) Let a, b, c and d be real numbers such that $\log_a(b) = 6$, $\log_b(c) = \frac{1}{2}$ and $\log_c(d) = 3$.

Find the value of $\log_d\left(\sqrt{abc}\right)$. Express your answer as a rational number in lowest terms.

$$\log_a(b) = 6 \Longrightarrow a^6 = b \Longrightarrow a = b^{\frac{1}{6}}$$

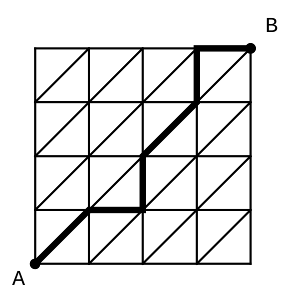
$$\log_b(c) = \frac{1}{2} \Longrightarrow b^{\frac{1}{2}} = c \Longrightarrow b = c^2 \Longrightarrow a = c^{\frac{1}{3}}$$

$$abc = c^{\frac{1}{3}}\cdot c^2\cdot c = c^{\frac{10}{3}}$$

$$\log_c(d) = 3 \Longrightarrow \log_d(c) = \frac{1}{3}$$

$$\log_d\sqrt{abc} = \log_d\left(c^{\frac{10}{3}\cdot\frac{1}{2}}\right) = \log_d c^{\frac{5}{3}} = \frac{5}{3}\log_d(c) = \frac{5}{3}\cdot\frac{1}{3} = \mathbf{\frac{5}{9}}$$

24) How many different paths are there from point A to point B, travelling upward, to the right, or diagonally upward along lines in the figure? One such path is illustrated.



We can represent each path as a sequence of moves, U = up, R = right or D = diagonal. For example, the illustrated path would be DRUDUR. We can see that the number of D's could be 0, 1, 2, 3 or 4. We will specify the number of D's, place the D's then place the U's. Note that each D decreases both the number of U's and the number of R's by 1.

0 D's _ _ _ _ _ _ _ _ $\binom{8}{0}\cdot\binom{8}{4} = 70$

1 D _ _ _ _ _ _ _ $\binom{7}{1}\cdot\binom{6}{3} = 140$

2 D's _ _ _ _ _ _ $\binom{6}{2} \cdot \binom{4}{2} = 90$

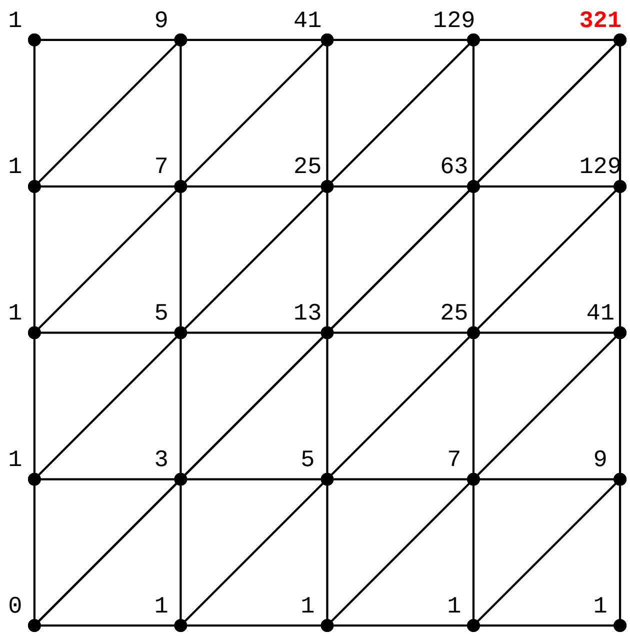
3 D's _ _ _ _ _ $\binom{5}{3} \cdot \binom{2}{1} = 20$

4 D's _ _ _ _ $\binom{4}{4} = 1$

Number of paths = 70+140+90+20+1 = **321**

OR

We can label each node with the number of ways to reach that node:

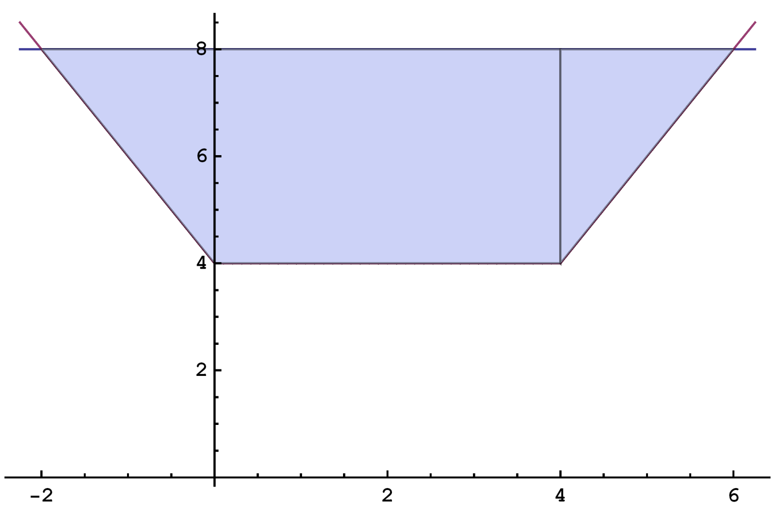


25) Find the area of the region in the plane consisting of all points (x,y) such that $|x| + |x-4| \le y \le 8$.

For $x \le 0$ $|x| + |x-4| = -x + 4 - x = 4 - 2x$ $4 - 2x = 8 \implies x = -2$

For $0 \le x \le 4$ $|x| + |x-4| = x + 4 - x = 4$

For $x \ge 4$ $|x| + |x-4| = x + x - 4 = 2x - 4$ $2x - 4 = 8 \implies x = 6$



Area = $\frac{1}{2} \cdot 4 \cdot (4 + 8) =$ **24**

26) On three consecutive days, Tom left his house at 7:30 in the morning to drive to work. On the first day, he drove at 40 mph and arrived 3 minutes late. On the second day, he drove at 60 mph and arrived 8 minutes early. On the third day, he arrived exactly on time. How fast did Tom drive on the third day?

Let t = on-time time, d = distance and r = third day rate.

$$\left(t + \frac{3}{60}\right) 40 = d$$

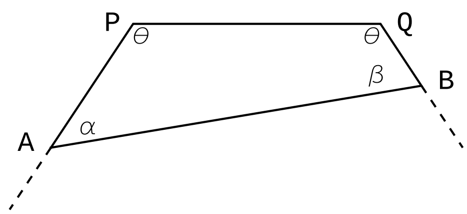
$$\left(t - \frac{8}{60}\right) 60 = d$$

$$\left(t - \frac{8}{60}\right) 60 = \left(t + \frac{3}{60}\right) 40 \implies 60t - 8 = 40t + 2 \implies 20t = 10 \implies t = \frac{1}{2}$$

$$\left(\frac{1}{2} + \frac{1}{20}\right) 40 = d \implies d = 22$$

$$r = \frac{22}{\frac{1}{2}} = \mathbf{44}$$

27) The figure displays a corner of a regular polygon that contains two consecutive vertices, P and Q. The corner was removed from the polygon by cutting along the line segment from A to B, where A and B are points on the boundary of the polygon. It is known that $\alpha + \beta = 40°$, where α and β are the respective angles at A and B. How many sides does the polygon have?



$$\alpha + \beta = 40° \implies \theta = \frac{360 - 40}{2} = 160°$$

If n = number of sides, $160 = \frac{(n-2)(180)}{n}$

$$160n = 180n - 360 \implies 20n = 360 \implies n = \frac{360}{20} = \mathbf{18}$$

28) Suppose that $f(x) = x^4 + a x^3 + b x^2 + c x + d$ where a, b, c and d are real numbers. If $f(1+2i) = f(3i) = 0$, where $i^2 = -1$, what is the value of a – b + c – d ?

$$f(x) = (x - (1+2i))(x - (1-2i))(x - 3i)(x + 3i)$$

$$f(x) = x^4 + a x^3 + b x^2 + c x + d \implies f(-1) = 1 - a + b - c + d$$

$$f(-1) = (-1 - (1+2i))(-1 - (1-2i))(-1 - 3i)(-1 + 3i) = (-2 - 2i)(-2 + 2i))(-1 - 3i)(-1 + 3i)$$

$$f(-1) = (4 + 4)(1 + 9) = 8(10) = 80$$

$$f(-1) = 1 - a + b - c + d = 80 \implies a - b + c - d = \mathbf{-79}$$

29) If k is a positive integer, let S(k) be the sum of the digits of k. For example, S(322) = 7 and S(8) = 8. For how many two-digit positive integers n is S(S(n)) = 5 ?

$10 \le n \le 99 \implies 1 \le f(n) \le 18$ and f(f(n)) = 5 f(n) = 5 or 14

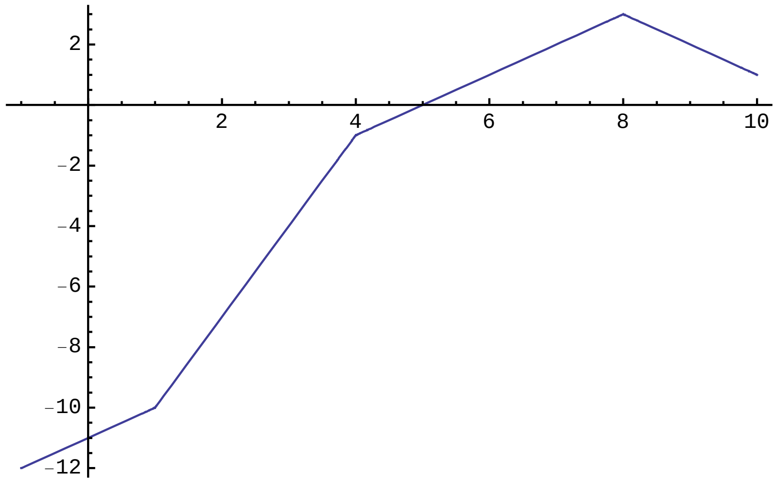
If f(n) = 5, then n = 50, 14, 41, 23 or 32   5 times

If f(n) = 14, then n = 59, 95, 68, 86 or 77   5 times

Total number of n = 5 + 5 = **10**

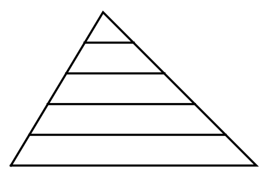
30) For what real x is f(x) = | x – 1 | – | x – 4 | – | x – 8 | a maximum ?

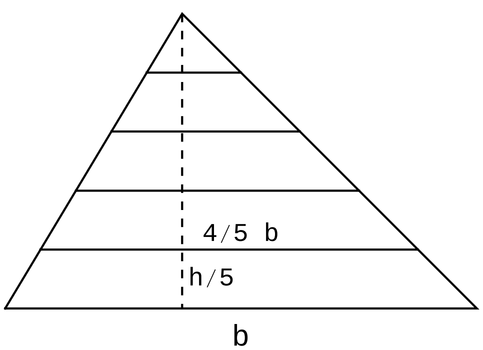
|x − 8|	8 − x	8 − x	8 − x	x − 8
|x − 4|	4 − x	4 − x	x − 4	x − 4
|x − 1|	1 − x	x − 1	x − 1	x − 1
	1−x−(4−x)−(8−x)	x−1−(4−x)−(8−x)	x−1−(x−4)−(8−x)	x−1−(x−4)−(x−8)
	1−x−4+x−8+x	x−1−4+x−8+x	x−1−x+4−8+x	x−1−x+4−x+8
	x−11	3x−13	x−5	−x+11



Maximum when x = 8 Value = **3**

31) Four lines parallel to the base of a triangle divide the other two sides of the triangle into five equal segments and the triangle into five distinct regions. If the area of the largest of these regions is 27, what is the area of the triangle?

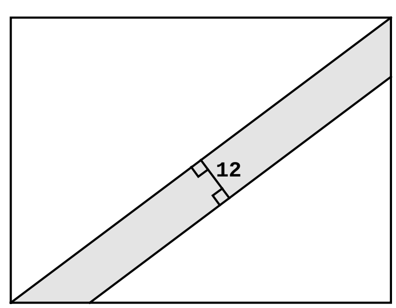


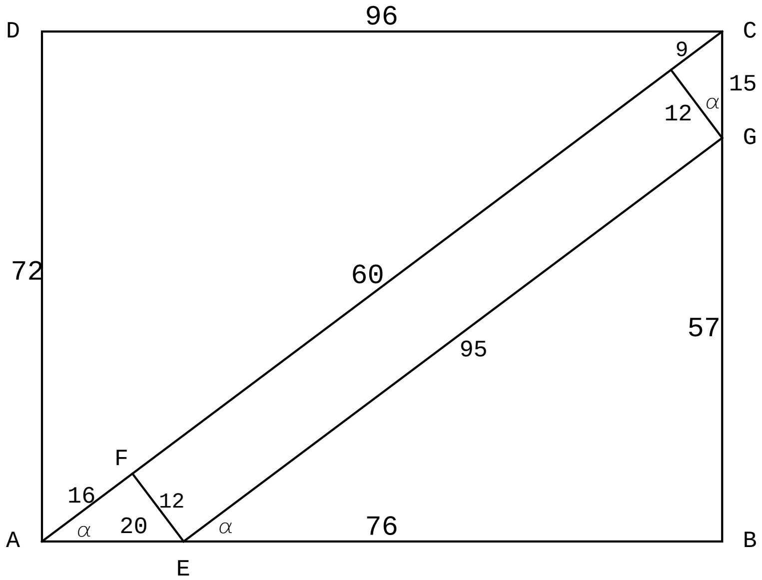


$A_{Tr} = \frac{1}{2}\left(\frac{h}{5}\right)\left(b + \frac{4}{5}\,b\right)$

$27 = \frac{1}{10}\,h\left(\frac{9}{5}\,b\right) = \frac{1}{2}bh \times \frac{9}{25} \implies \frac{1}{2}\,bh = 27 \times \frac{25}{9} = \mathbf{75}$

32) A 72 x 96 foot rectangular grass plot is cut diagonally by a 12 foot wide paved walkway. If one edge of the walkway is along the diagonal of the rectangular plot, what is the area of the paved walkway? The walkway is the shaded region in the sketch.





From ΔABC $\tan(\alpha) = \frac{72}{96} = \frac{3}{4} \implies \sin(\alpha) = \frac{3}{5}$

From ΔAEF $\sin(\alpha) = \frac{3}{5} = \frac{12}{AE} \implies AE = 12\left(\frac{5}{3}\right) = 20 \implies EB = 76$

From ΔEBG $\tan(\alpha) = \frac{3}{4} = \frac{BG}{76} \implies BG = 76\left(\frac{3}{4}\right) = 57$

Area of walkway = area ΔABC – area Δ EBG = $\frac{1}{2}(96)(72) - \frac{1}{2}(76)(57)$

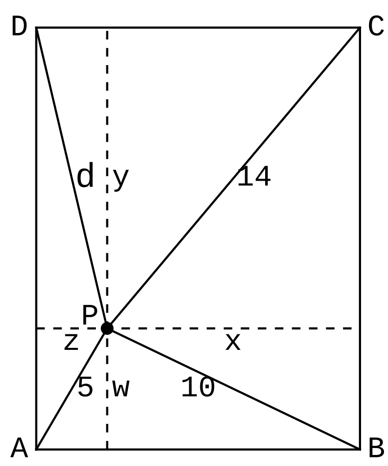
Area of walkway = 3456 –2166 = **1290**

33) If $0° < \theta < 90°$ and $\sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{5}}$, find $\tan(\theta)$.

$$\cos(\theta) = \cos\left(2\left(\frac{\theta}{2}\right)\right) = 1 - \sin^2\left(\frac{\theta}{2}\right) = 1 - \frac{1}{5} = \frac{4}{5} \implies \sin(\theta) = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\tan(\theta) = \frac{\frac{3}{5}}{\frac{4}{5}} = \mathbf{\frac{3}{4}}$$

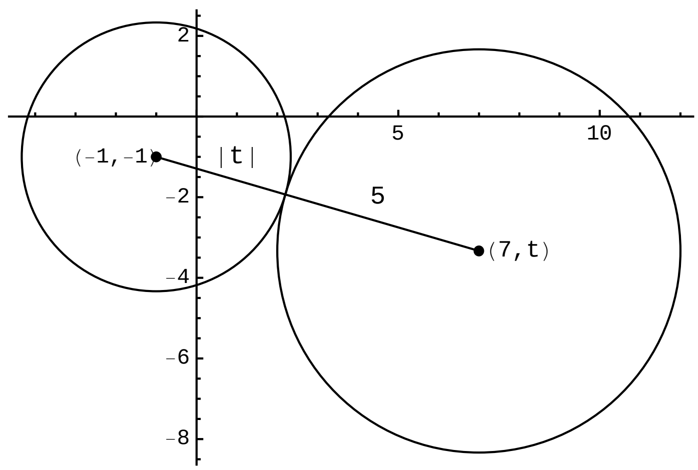
34) Let P be a point in the interior of a rectangle ABCD. If AP = 5, BP = 10 and CP = 14, find the length DP.



$$\begin{cases} z^2 + w^2 = 5^2 & (1) \\ x^2 + w^2 = 10^2 & (2) \\ x^2 + y^2 = 14^2 & (3) \\ z^2 + y^2 = d^2 & (4) \end{cases}$$

(1) + (3) = (2) + (4) $\Longrightarrow$ $5^2 + 14^2 = 10^2 + d^2$ $\Longrightarrow$ $d^2 = 121$ $\Longrightarrow$ **d = 11**

35) For what values of t are the circles defined by the equations $\begin{cases} (x+1)^2 + (y+1)^2 = t^2 \\ (x-7)^2 + (y-t)^2 = 5^2 \end{cases}$ tangent?

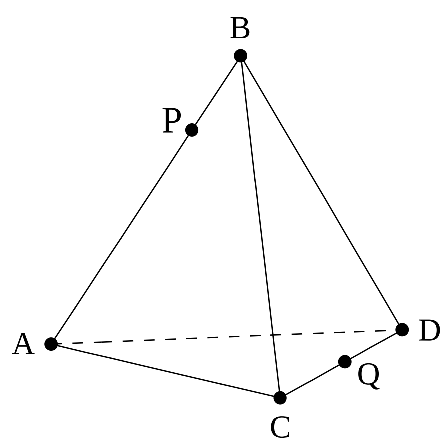


If t > 0, $8^2 + (t+1)^2 = (5+t)^2$ $\Longrightarrow$ $64 + t^2 + 2t + 1 = 25 + 10t + t^2$ $\Longrightarrow$ 8t = 40 ==> t = 5

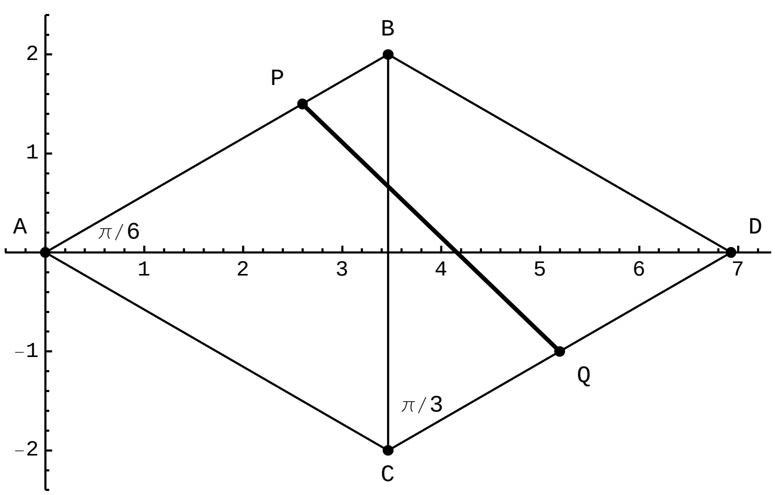
If t < 0, $8^2 + (t+1)^2 = (5-t)^2$ $\Longrightarrow$ $64 + t^2 + 2t + 1 = 25 - 10t + t^2$ $\Longrightarrow$ 12t = – 40 ==> $t = -\frac{10}{3}$

$t = -\frac{10}{3}$ or 5

36) Each edge of the tetrahedron ABCD has length 4. Point P is on edge $\overline{AB}$ and AP = 3. Point Q is the midpoint of edge $\overline{CD}$. A bug travels from P to Q, always moving on the surface of the tetrahedron. What is the shortest distance that the bug can travel?



Flatten faces ABC and BCD along the line BC. Place the origin at A.

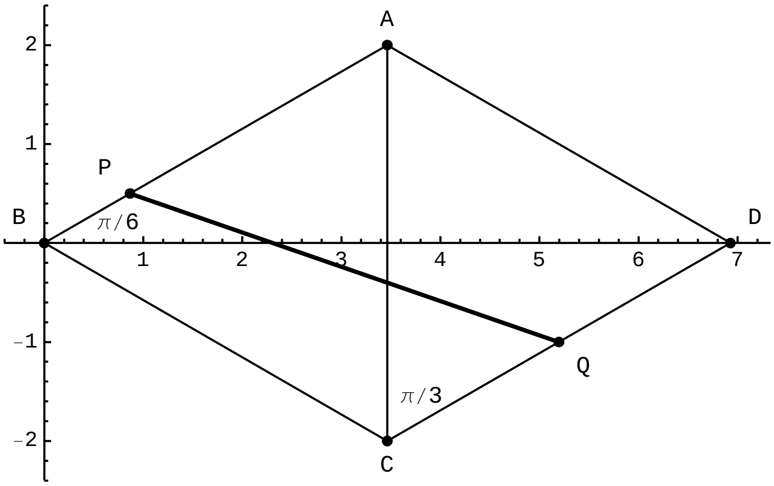


$P = 3\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $C = 4\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $Q = C + 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(3\sqrt{3}, -1\right)$

$PQ^2 = \left(\frac{3}{2}\sqrt{3} - 3\sqrt{3}\right)^2 + \left(\frac{3}{2} + 1\right)^2 = \left(-\frac{3}{2}\sqrt{3}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{27}{4} + \frac{25}{4} = \frac{52}{4} = 13$

$PQ = \sqrt{13}$

Flatten faces ABC and ACD along the line AC. Place the origin at B.

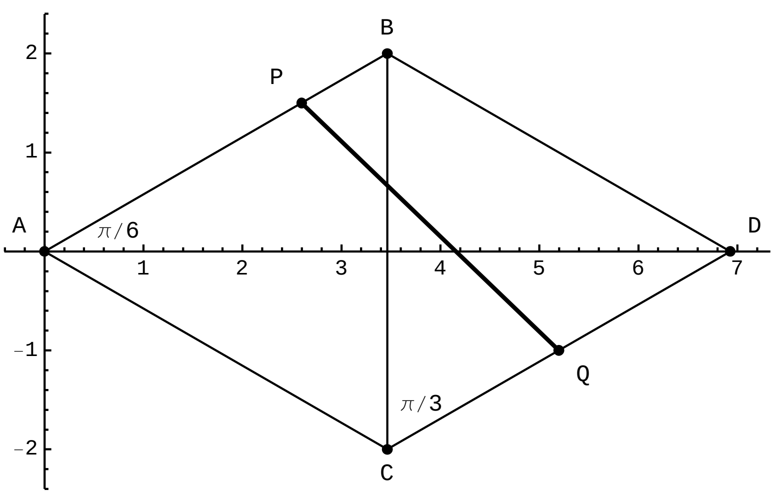


$P = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $C = 4\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $Q = C + 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(3\sqrt{3}, -1\right)$

$PQ^2 = \left(\frac{1}{2}\sqrt{3} - 3\sqrt{3}\right)^2 + \left(\frac{1}{2} + 1\right)^2 = \left(-\frac{5}{2}\sqrt{3}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{75}{4} + \frac{9}{4} = \frac{84}{4} = 21$

$PQ = \sqrt{21}$

Flatten faces ABD and ACD along the line AD. Place the origin at A.



$P = 3\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $C = 4\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $Q = C + 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(3\sqrt{3}, -1\right)$

$PQ^2 = \left(\frac{3}{2}\sqrt{3} - 3\sqrt{3}\right)^2 + \left(\frac{3}{2} + 1\right)^2 = \left(-\frac{3}{2}\sqrt{3}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{27}{4} + \frac{25}{4} = \frac{52}{4} = 13$

$PQ = \sqrt{13}$

Thus the minimum distance is $\mathbf{\sqrt{13}}$.

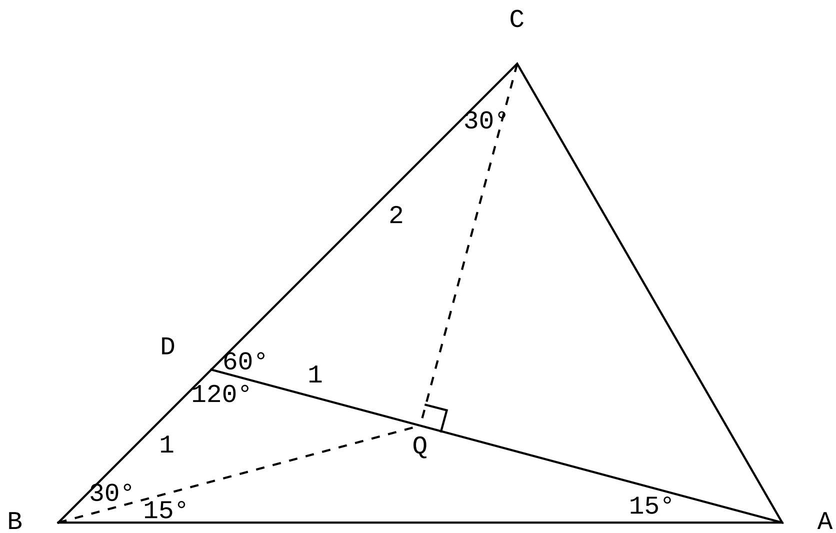
37) How many ordered pairs (a , b) of real numbers satisfy the equation $(a + b\,i)^{2010} = a - b\,i$, where $i^2 = -1$?

Let $z = a + b\,i$ $\quad |z| = |\overline{z}| \implies |z|^{2010} = |z| \implies |z|\left(|z|^{2009} - 1\right) = 0 \implies |z| = 0$ or 1

$z^{2010} = \overline{z} \implies z \cdot z^{2010} = \overline{z} \cdot z = 1 \implies z^{2011} = 1$ which has 2011 solutions.

$|z| = 0$ has one solution $\implies$ Total number of solutions to $z^{2010} = \overline{z}$ is 2011 + 1 = **2012**

38) In triangle ABC, $\angle ABC = 45°$ and BC = 3.
Point D is on $\overline{BC}$ and BD = 1. If $\angle DAB = 15°$,
what is the degree measure of $\angle BAC$?



Draw a perpendicular from C meeting DA at point Q. Connect B to Q.

From ΔBDA $\angle BDA = 120° \implies \angle CDQ = 60° \implies \angle DCQ = 30°$

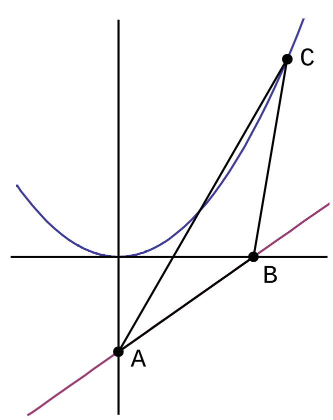
From right triangle DQA DQ = 1 $\implies$ ΔBDQ is isosceles $\implies \angle DBQ = \frac{180° - 120°}{2} = 30°$.

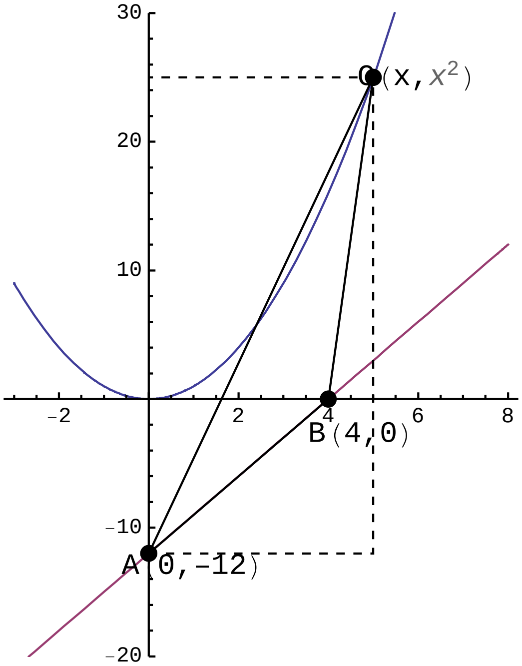
Thus $\angle QBA = 15° \implies$ ΔBQA is isosceles $\implies$ BQ = QA

ΔBQC is also isosceles $\implies$ BQ = QC. Thus QA = QC $\implies$ ΔCQA is isosceles $\implies$
$\angle QCA = \angle QAC = \frac{90°}{2} = 45°$

Thus $\angle BAC = 15° + 45° =$ **60°**

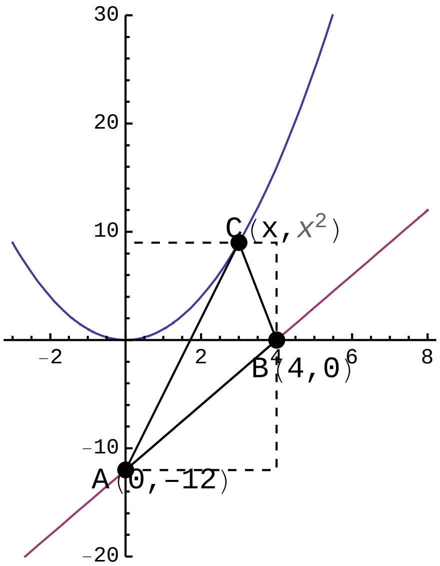
39) Let A = (0, – 12) and B = (4 , 0) be points in the plane. If C is a point on the parabola $y = x^2$, find x so that the area of triangle ABC is a minimum.





$x \geq 4$ $\quad \frac{1}{2}x\left(x^2+12\right) - \frac{1}{2}(x-4)\,x^2 - \frac{1}{2}(12)(x+x-4) = \frac{1}{2}x^3 + 6x - \frac{1}{2}x^3 + 2x^2 - 12x + 24$

$= 2x^2 - 6x + 24$



$x < 4$ $\quad 4\left(x^2+12\right) - \frac{1}{2}\left(x^2+12\right)x - \frac{1}{2}(4-x)\left(x^2\right) - \frac{1}{2}\,4\,(12) = 4x^2 + 48 - \frac{1}{2}x^3 - 6x - 2x^2 + \frac{1}{2}x^3 - 24$

$= 2x^2 - 6x + 24$

$2x^2 - 6x + 24 = 2\left(x^2 - 3x + 12\right) = 2\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 12\right) = 2\left(\left(x - \frac{3}{2}\right)^2 + \frac{39}{4}\right)$

Minimum when x = $\frac{3}{2}$.

40) If the sum of a list of an odd number of consecutive positive integers is 2010, what is the smallest possible integer in any such list?

$n + (n+1) + (n+2) + (n+3) + \cdots + (n+2k) = 2010$

$(2k+1) \cdot n + 1 + 2 + 3 + \cdots + 2k = 2010$

$(2k+1) \cdot n + \frac{(2k)(2k+1)}{2} = 2010$

$(2k + 1)(n + k) = 2010$

$2010 = 2 \cdot 3 \cdot 5 \cdot 67$

$2k + 1$ odd $\Longrightarrow$

$2k + 1 = 3 \Longrightarrow k = 1 \Longrightarrow 3(n+1) = 2010 \Longrightarrow n = 669$

$2k + 1 = 5 \Longrightarrow k = 2 \Longrightarrow 5(n+2) = 2010 \Longrightarrow n = 400$

$2k + 1 = 15 \Longrightarrow k = 7 \Longrightarrow 7(n+7) = 2010 \Longrightarrow n = 127$

$2k + 1 = 67 \Longrightarrow k = 33 \Longrightarrow 33(n+67) = 2010 \Longrightarrow n = -3$

Since n decreases as k increases, the smallest positive first integer is **n = 127.**

41) Find the value of $\sum_{k=1}^{10} \frac{k}{k^4 + k^2 + 1}$. Express your answer as a rational number in lowest terms.

$$k^4 + k^2 + 1 = k^4 + 2k^2 + 1 - k^2 = (k^2 + 1)^2 - k^2 = (k^2 + 1 - k)(k^2 + 1 + k)$$

$$\frac{1}{k^2+k-1} - \frac{1}{k^2+k+1} = \frac{k^2+k-1-(k^2+k-1)}{k^4+k^2+1} = \frac{2k}{k^4+k^2+1}$$

Let $f(k) = \frac{1}{k^2+k+1} \Longrightarrow f(k-1) = \frac{1}{k^2-k+1}$

$$\sum_{k=1}^{a} \frac{k}{k^4+k^2+1} = \frac{1}{2}\sum_{k=1}^{a} [f(k-1) - f(k)] = \frac{1}{2}[f(0) - f(a)]$$

$$= \frac{1}{2}\sum_{k=1}^{a} [1 - \frac{1}{a^2+a+1}] = \frac{1}{2}\left[\frac{a^2+a+1-1}{a^2+a+1}\right] = \frac{1}{2}\left[\frac{a^2+a}{a^2+a+1}\right]$$

$$\sum_{k=1}^{10} \frac{k}{k^4+k^2+1} = \frac{1}{2}\left[\frac{100+10}{100+10+1}\right] = \frac{110}{2(111)} = \frac{55}{111}$$