

EE -171 SIGNALS AND SYSTEMS
FINAL EXAM
December 16, 2010
7:30 a.m. - 10:15 a.m.

Do all problems.

SOLUTIONS

NAME _____

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Formulae:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\Omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\Omega_0 t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega) e^{j\omega_0 t} d\omega$$

$$X(\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m] h[n - m]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(\frac{2\pi}{N})n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$e^{-at} u(t) = x(t) \Leftrightarrow X(\omega) = \frac{1}{(j\omega + a)}$$

$$tx(t) \Leftrightarrow \frac{1}{(j\omega + a)^2}$$

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

$$\text{Cos}\omega_0 t \Leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

(Note: There are a total of 100 points.)

1. (Power signals and energy signals)

(10 points)

We distinguish between *power* signals and *energy* signals. Recall that an energy signal is one with finite energy, that is,

$$E = \int_{-\infty}^{+\infty} |s(t)|^2 dt < \infty$$

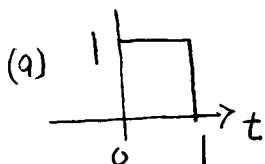
and a power signal is one where,

$$P_{ave} = \lim_{T \rightarrow \infty} \int_{T/2}^{T/2} |s(t)|^2 dt < \infty$$

So, *power signals* are signals with average power and *energy signals* are those with finite energy.

Do the following:

- Give an example of a signal with finite duration, that has finite energy.
- Give an example of a signal that decays exponentially to zero, that has finite energy.
- Give an example of a signal that is not an energy signal, but which is a power signal.
- Based on the above, how do the Fourier transforms of power and energy signals differ?



$$E = \int_0^1 1 dt = 1$$

(b) $x(t) = e^{-t} u(t)$

$$E = \int_0^{\infty} e^{-2t} dt = \frac{1}{-2} [0 - 1] = \frac{1}{2}$$

(c) $x(t) = \cos t$
 $\omega = 1 = \frac{2\pi}{T}$

$$E = \int_{-\infty}^{+\infty} |\sin(t)|^2 dt \neq \infty$$

$$P_{ave} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos^2 t dt = \frac{1}{2\pi} \frac{1}{2} \int_{-\pi}^{+\pi} (1 + \cos 2t) dt$$

$$= \frac{1}{4\pi} \left[t + \frac{1}{2} \sin 2t \right]_{-\pi}^{+\pi} = \frac{1}{4\pi} [\pi - (-\pi)] = \frac{1}{2}$$

(d) FT (energy signal) exist in the usual sense.
 FT (power signal) exists in the "generalized sense".

2. (Periodic signals)
(5 points)

Determine whether the following signals are periodic:

(a) $s_1(t) = e^{\sin(t)}$

(b) $s_2(t) = e^{t \sin(t)}$

(a) $s_1(t) = e^{\sin t} \stackrel{?}{=} e^{\sin(t+T)}$ Yes, when $T=2\pi$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

(b) $s_2(t) = e^{t \sin t} \stackrel{?}{=} e^{(t+T) \sin(t+T)}$ No.

3. (Differential equations)

(15 points) Consider the first order linear time-invariant system given by,

$$\frac{dy}{dt} + 4y(t) = x(t) \tag{1}$$

where the system is initially at rest.

- (a) When the input $x(t) = e^{-t}u(t)$ determine the total solution $y(t)$ for all $t \geq 0$.
- (b) For the case where the input is $x(t) = \delta(t)$, show that the impulse response is

$$h(t) = e^{-4t}u(t).$$

(For initial conditions, in order to determine value of $y(t)$ immediately after application of the unit impulse, consider integrating (1) from $t = 0^-$ to $t = 0^+$. Hence determine $y(0^+)$. Assume $\int_{0^-}^{0^+} y(t)dt = 0$.)

- (c) Using the impulse response of (b) and input $x(t)$ of (a), determine $y(t) = h(t) * x(t)$ and compare with $y(t)$ found in (a).

4. (Difference equations)

(15 points)

(a) Consider the discrete LTI system represented by

$$y[n] = x[n] - x[n - 1]$$

where the system is initially at rest. Determine impulse response $h[n]$. Plot it. Is the system stable?

(b) Now consider the discrete LTI system given by

$$y[n] = y[n - 1] + x[n]$$

where the system is initially at rest. Determine the impulse response $h[n]$. Plot it. Is the system stable?

(c) For the system in (b), determine step response $y[n]$ when $x[n] = u[n]$. Plot it.

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3. (Differential equations)

Consider the first order linear time-invariant system given by.

$$\frac{dy}{dt} + 4y(t) = x(t) \quad (1)$$

where the system is initially at rest.

- (a) When the input $x(t) = e^{-t}u(t)$ determine the total solution $y(t)$ for all $t \geq 0$.
 (b) For the case where the input is $x(t) = \delta(t)$, show that the impulse response is

$$h(t) = e^{-4t}u(t).$$

(For initial conditions, in order to determine value of $y(t)$ immediately after application of the unit impulse, consider integrating (1) from $t = 0^-$ to $t = 0^+$. Hence determine $y(0^+)$. Assume $\int_{0^-}^{0^+} y(t) dt = 0$.)

- (c) Using the impulse response of (b) and input $x(t)$ of (a), determine $y(t) = h(t) * x(t)$ and compare with $y(t)$ found in (a).

$$(a) \quad \frac{dy}{dt} + 4y = x(t)$$

$$\begin{aligned} \underline{y_H(t)}: \quad \frac{dy_H(t)}{dt} + 4y_H(t) &= 0 & \text{Try } y_H(t) &= Ke^{st} \\ \therefore sKe^{st} + 4Ke^{st} &= 0 & Ke^{st}(s+4) &= 0 \quad s = -4 \\ \therefore y_H(t) &= Ke^{-4t} \end{aligned}$$

$$\underline{y_P(t)} \quad \frac{dy_P(t)}{dt} + 4y_P(t) = e^{-t} \quad t > 0$$

$$\begin{aligned} \text{Try } y_P(t) &= Ae^{-t} \\ \therefore -Ae^{-t} + 4Ae^{-t} &= e^{-t} & \therefore 3Ae^{-t} &= e^{-t}, \text{ Yes, if } A = \frac{1}{3} \\ \therefore y_P(t) &= \frac{1}{3}e^{-t} \end{aligned}$$

$$\underline{y_{TOTAL}} = y_H(t) + y_P(t) = Ke^{-4t} + \frac{1}{3}e^{-t}$$

$$y(0) = 0 = K + \frac{1}{3} \Rightarrow K = -\frac{1}{3}$$

$$\therefore y(t) = \left(-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} \right) u(t) \quad (2)$$



$$(b) \frac{dy}{dt} - 4y(t) = \delta(t)$$

$$\text{For } t > 0 \quad \frac{dy}{dt} + 4y(t) = 0 \Rightarrow y(t) = Ae^{-4t}$$

$$\text{I.C.} \quad \int_{0^-}^{0^+} \frac{dy}{dt} dt + \int_{0^-}^{0^+} 4y(t) dt = \int_{0^-}^{0^+} \delta(t) dt$$

$$\therefore y(0^+) - y(0^-) + 0 = 1 \quad \therefore y(0^+) = 1$$

$$\therefore \text{apply it, I.C. } y(0^+) = 1 = Ae^0 \Rightarrow A = 1$$

$$\therefore \boxed{h(t) = e^{-4t} u(t)} \quad (b)$$

$$(c) y(t) = h(t) * x(t) = e^{-4t} u(t) * e^{-t} u(t)$$

$$= \int_{-\infty}^{+\infty} e^{-4\tau} u(\tau) e^{-(t-\tau)} u(t-\tau) d\tau = \int_0^t e^{-3\tau} d\tau e^{-t} \quad t \geq 0$$

$$= e^{-t} \left[\frac{1}{-3} (e^{-3\tau}) \Big|_0^t \right] = \frac{e^{-t}}{-3} (e^{-3t} - 1)$$

$$\boxed{y(t) = \left(\frac{e^{-4t}}{-3} + \frac{1}{3} e^{-t} \right) u(t)} \quad (c)$$

15 4. (Difference equations)

(a) Consider the discrete LTI system represented by

$$y[n] = x[n] - x[n-1]$$

where the system is initially at rest. Determine impulse response $h[n]$. Plot it. Is the system stable?

(b) Now consider the discrete LTI system given by

$$y[n] = y[n-1] + x[n]$$

where the system is initially at rest. Determine the impulse response $h[n]$. Plot it. Is the system stable?

(c) For the system in (b), determine step response $y[n]$ when $x[n] = u[n]$. Plot it.

(a) $y[n] = x[n] - x[n-1]$ $x[n] = \delta[n]$

$\therefore h[n] = \delta[n] - \delta[n-1]$

$\sum_{n=-\infty}^{\infty} |h[n]| = |h[0]| + |h[1]| = 2 < \infty$ **STABLE**

(b) $y[n] = y[n-1] + x[n]$ $x[n] = \delta[n]$

For $n > 0$ $y[n] - y[n-1] = 0$

Try $y[n] = A\alpha^n$ $\therefore A\alpha^n - A\alpha^{n-1} = 0$

i.e. $A\alpha^n [1 - \alpha^{-1}] = 0$ Yes, if $\alpha = 1$

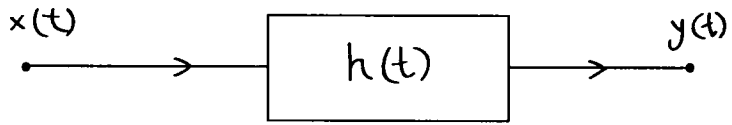
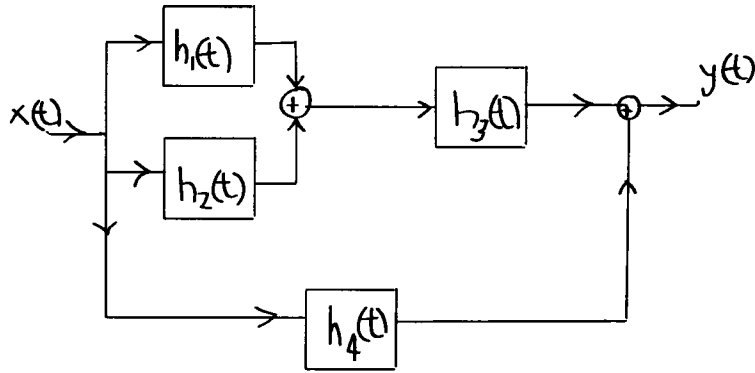
$\therefore y[n] [h[n]] = A1^n \quad n > 0$

$y[0] = y[-1] + 1 \quad \therefore y[0] = 1$ Using $\sum_{n=0}^{\infty} \delta[n] = 1 \quad n \geq 0$ (b)

$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 1 = \infty$ **UNSTABLE**

(c) $y[n] = u[n] + u[n-1] = \sum_{m=0}^n u[m] = (n+1)u[n]$ (c)

5. (Connected systems)
(5 points)

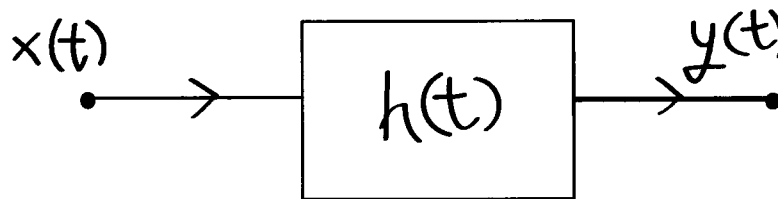


Do the following:

- (a) Determine the total impulse response $h(t)$ of the connected system.

$$h(t) = (h_1(t) + h_2(t)) * h_3(t) + h_4(t)$$

6. (Continuous-time convolution)
(10 points)



(i) Assume that the input-output relationship for the system shown is,

$$y(t) = \int_{-\infty}^t e^{-2(t-\tau)} x(\tau - 1) d\tau$$

Do the following:

- (a) Find the impulse response of the system.
- (b) Is the system causal? Explain.
- (c) Is the system stable? Explain.

(ii) Assume now that the input-output relationship for the system shown is

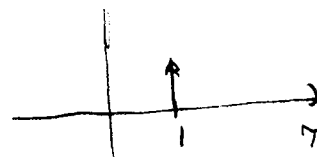
$$y(t) = \int_{-\infty}^{+\infty} e^{-2(t-\tau)} x(\tau - 1) d\tau$$

Do the following:

- (d) Find the impulse response of the system.
- (e) Is the system causal? Explain.
- (f) Is the system stable? Explain.

$$\begin{aligned} \text{(a)} \quad y(t) &= \int_{-\infty}^t e^{-2(t-\tau)} x(\tau-1) d\tau \Rightarrow h(t) = \int_{-\infty}^t e^{-2(t-\tau)} \delta(\tau-1) d\tau \\ &= \int_{-\infty}^t e^{-2(t-\tau)} \delta(\tau-1) d\tau = e^{-2(t-1)} \int_{-\infty}^t \delta(\tau-1) d\tau \end{aligned}$$

$$\boxed{h(t) = e^{-2(t-1)} u(t-1)}$$



(b) Causal

(c) Stable.



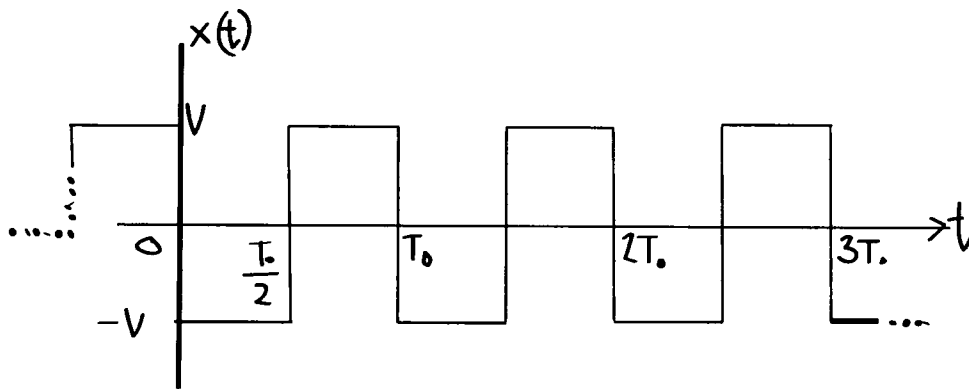
$$d) h(t) = \int_{-\infty}^{+\infty} e^{-2(t-\tau)} \delta(\tau-1) d\tau$$

$$\boxed{h(t) = e^{-2(t-1)} \int_{-\infty}^{+\infty} \delta(\tau-1) d\tau = e^{-2(t-1)}}$$

e) not causal

f) " stable since $\int |h(t)| dt < \infty$

7. (Continuous-time Fourier series)
(10 points)



The square wave $x(t)$ is,

$$x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T_0 = 2\pi$$

Find the Fourier series coefficients \tilde{c}_k .

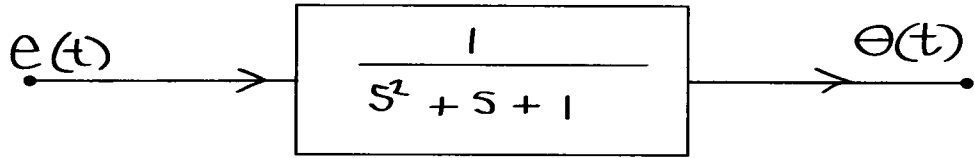
$$\begin{aligned} \tilde{c}_k &= \frac{1}{T_0} \int_0^{T_0/2} (-V) e^{-jk\omega_0 t} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} V e^{jk\omega_0 t} dt \\ &= \frac{-V}{T_0} \left(\frac{1}{-jk\omega_0} \right) \left[e^{jk\omega_0 t} \right]_0^{T_0/2} + \frac{V}{T_0} \left(\frac{1}{jk\omega_0} \right) \left[e^{-jk\omega_0 t} \right]_{T_0/2}^{T_0} \\ &= \frac{V}{jk\omega_0 T_0} \left[e^{-jk\omega_0 T_0/2} - 1 \right] - \frac{V}{T_0 jk\omega_0} \left[e^{-jk\omega_0 T_0} - e^{-jk\omega_0 T_0/2} \right] \end{aligned}$$

$$= \frac{V}{jk2\pi} \left[e^{-jk\pi} - 1 - e^{-jk2\pi} + e^{-jk\pi} \right]$$

$$\boxed{\tilde{c}_k} = \frac{V}{jk2\pi} \left[2e^{-jk\pi} - 2 \right] = \frac{V}{jk\pi} \left[e^{-jk\pi} - 1 \right]$$

8. (Laplace Transforms)
(10 points)

Servomotor



The common model of a servomotor is a second order model which often is of adequate accuracy. The input to the motor is the armature voltage $e(t)$ and the output signal is the motor shaft angle $\theta(t)$.

As shown in the figure, the input-output is expressed as,

$$\theta(s) = \frac{1}{s^2 + s + 1} e(s)$$

where $\theta(s)$ and $e(s)$ are the Laplace transforms of $\theta(t)$ and $e(t)$ respectively and $H(s) = \frac{1}{s^2 + s + 1}$ for $\text{Re}\{s\} > -1/2$.

We know that,

$$\frac{1}{s^2 + s + 1} \leftrightarrow e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t u(t)$$

It is desired to move the motor shaft a certain angle. That is done by applying a step signal for the motor armature voltage $e(t)$. Do the following:

For $e(t) = u(t)$, find and *sketch* $\theta(t)$.

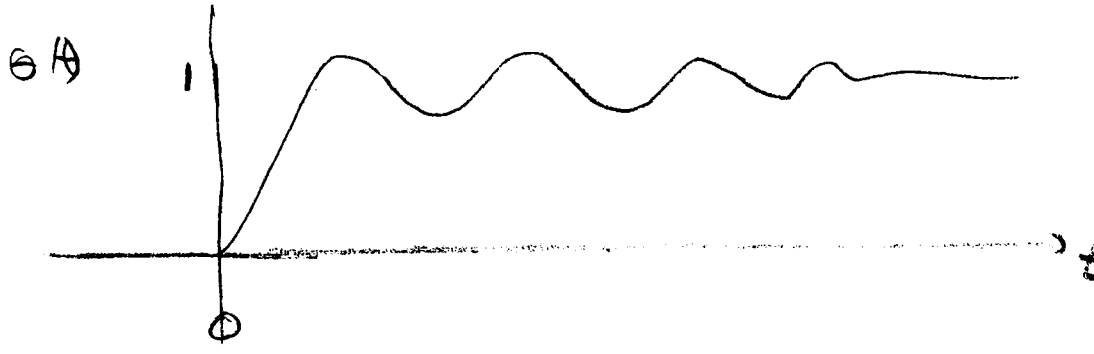
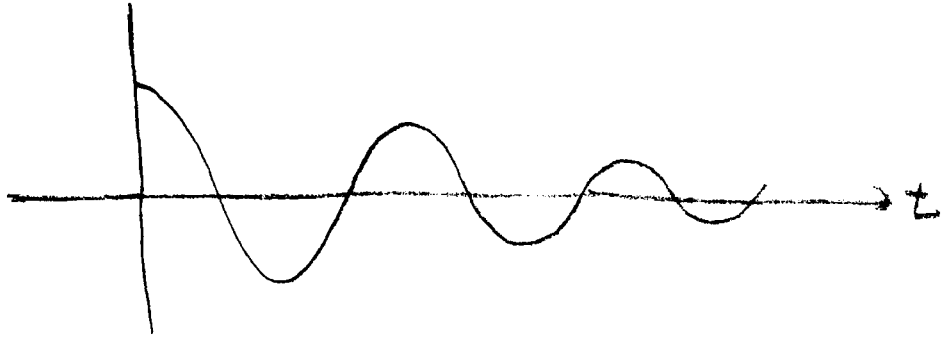
Note that you can solve the problem in the frequency domain using inverse Laplace transforms or work in the time domain by thinking in terms of the time-domain operation corresponding to $\frac{1}{s}$.

$$\theta(s) = \frac{1}{s^2 + s + 1} \cdot \frac{1}{s} = \frac{a}{s^2 + s + 1} + \frac{b}{s} \dots$$

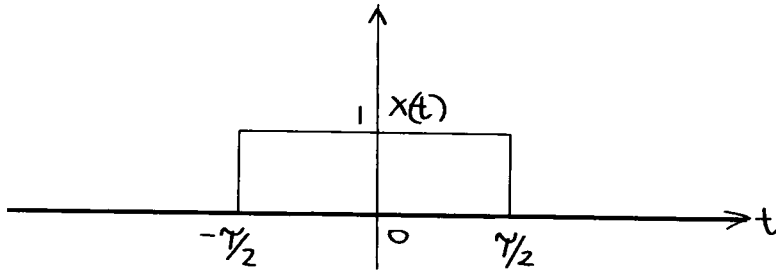


$$\Rightarrow \theta(t) = \int_0^t e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t dt$$





9. (Bandwidth of a rectangular pulse)
(10 points)

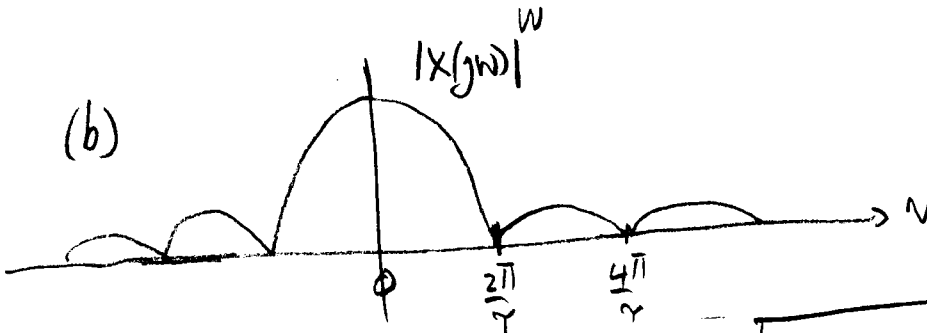


Rectangular pulses are used in many applications. It is desired to determine the bandwidth of the pulse shown. Do the following:

- Determine the Fourier transform $X(j\omega)$.
- Plot the amplitude spectrum $|X(j\omega)|$.
- Assuming that bandwidth is defined as the distance in frequency from $\omega = 0$ to the frequency where the amplitude spectrum first goes to zero, what is the bandwidth of this pulse?

$$\begin{aligned}
 \text{(a) } X(j\omega) &= \int_{-T/2}^{T/2} 1 e^{-j\omega t} dt = \frac{1}{-j\omega} \left[e^{-j\omega T/2} - e^{j\omega T/2} \right] \\
 &= \frac{1}{j\omega} \left[e^{j\omega T/2} - e^{-j\omega T/2} \right] = \frac{2}{\omega} \frac{(e^{j\omega T/2} - e^{-j\omega T/2})}{2j} \\
 &= 2 \frac{\sin \omega T/2}{\omega}
 \end{aligned}$$

$$\begin{aligned}
 \omega T/2 &= \pi k \\
 \omega &= \frac{2\pi}{T} k
 \end{aligned}$$



(c) $BW = 2\pi f_B = \frac{2\pi}{T}$

$$f_B = \frac{1}{T}$$

10. (Fourier Transforms)
(10 points)

Calculate the inverse Fourier transform of the following signals:

$$X(j\omega) = \frac{1}{j}[\delta(\omega - 1) - \delta(\omega + 1)] + 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi j} \int_{-\infty}^{+\infty} (\delta(\omega - 1) - \delta(\omega + 1)) e^{j\omega t} d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi j} (e^{jt} - e^{-jt}) + \frac{3}{2\pi} (e^{j2\pi t} + e^{-j2\pi t})$$

$$x(t) = \frac{1}{\pi} \sin t + \frac{3}{\pi} \cos 2\pi t$$