EE -171 SIGNALS AND SYSTEMS
FINAL EXAM
December 16, 2008
8:00 a.m. - 11:00 a.m.

Do all problems.
Each problem is 10 points.

NAME SOLUTIONS

1.____________

2.____________

3.____________

4.____________

5.____________

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7.____________

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9.____________

10.____________

Formulae:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\Omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{j\omega_0 t}d\omega$$

$$X(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m]$$

$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(\frac{2\pi}{N})n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$e^{-at}u(t) = x(t) \Leftrightarrow X(\omega) = \frac{1}{(j\omega+a)}$$

$$tx(t) \Leftrightarrow \frac{1}{(j\omega+a)^2}$$

$$x(at) \Leftrightarrow \frac{1}{|a|}X(\frac{j\omega}{a})$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$

$$Cos\omega_0 t \Leftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

(Note: There are a total of 100 points.)

1. (Signal power and signal energy)

Determine the (average) signal *power* of the following signals:

(a) $x(t) = A$

(b) $x(t) = \sin(t)$. {Recall: $cos^2(t) = \frac{1}{2}(1 + cos2t)$}

Determine the *energy* of the following signal:

(a) $x(t) = e^{-t}u(t)$.

(a) $$\boxed{P_{ave}} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 dt = \lim_{T\to\infty} \frac{1}{2T} A^2 t \Big|_{-T}^{T} = \lim_{T\to\infty} \frac{1}{2T} A^2(2T) = \boxed{A^2}$$

(b) $\sin(t+2\pi) = \sin t \quad \therefore T_0 = 2\pi$

$$\therefore P_{ave} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sin^2 t\, dt = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[1 - \frac{1}{2}(1+\cos 2t)\right]dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)dt = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{1}{2}\,dt - \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{1}{2}\cos 2t\, dt$$

$$P_{ave} = \frac{1}{2\pi}\left(\frac{1}{2}\right)\left[t\right]_{-\pi}^{+\pi} - 0 = \left(\frac{1}{2\pi}\right)\left(\frac{1}{2}\right)(2\pi) = \boxed{\frac{1}{2}}$$

(c) $$\text{Energy} = \int_{-\infty}^{+\infty} x^2(t)\,dt = \int_{-\infty}^{+\infty} e^{-2t}u(t)\,dt = \int_{0}^{\infty} e^{-2t}\,dt$$

$$= \frac{1}{-2}\left[e^{-2t}\right]_0^{\infty} = \frac{1}{-2}\left[0 - 1\right] = \boxed{\frac{1}{2}}$$

2. (Differential equations)

Consider the first order linear time-invariant system given by,

$$\frac{dy}{dt} + 4y(t) = x(t) \tag{1}$$

where the system is initially at rest.

(a) When the input $x(t) = e^{-t}u(t)$ determine the total solution $y(t)$ for all $t \geq 0$.

(b) For the case where the input is $x(t) = \delta(t)$, show that the impulse response is

$$h(t) = e^{-4t}u(t).$$

(For initial conditions, in order to determine value of y(t) immediately after application of the unit impulse, consider integrating (1) from $t = 0^-$ to $t = 0^+$. Hence determine $y(0^+)$. Assume $\int_{0^-}^{0^+} y(t)dt = 0$.

(c) Using the impulse response of (b) and input $x(t)$ of (a), determine $y(t) = h(t) * x(t)$ and compare with y(t) found in (a).

(a) $\frac{dy}{dt} + 4y = x(t)$

$y_H(t)$: $\frac{dy(t)}{dt} + 4y(t) = 0$ Try $y_H(t) = Ke^{st}$

$\therefore sKe^{st} + 4Ke^{st} = 0$ $Ke^{st}(s+4) = 0$ $s = -4$

$\therefore y_H(t) = Ke^{-4t}$

$y_P(t)$ $\frac{dy(t)}{dt} + 4y(t) = e^{-t}$ $t > 0$

Try $y_P(t) = Ae^{-t}$

$\therefore -Ae^{-t} + 4Ae^{-t} \stackrel{?}{=} e^{-t}$ $\therefore 3Ae^{-t} \stackrel{?}{=} e^{-t}$. Yes, if $A = \frac{1}{3}$

$\therefore y_P(t) = \frac{1}{3}e^{-t}$

$y_{TOTAL} = y_H(t) + y_P(t) = Ke^{-4t} + \frac{1}{3}e^{-t}$

$y(0) = 0 = K + \frac{1}{3} \Rightarrow K = -\frac{1}{3}$ $\boxed{\therefore y(t)_{TOTAL} = \left(-\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}\right)u(t)}$

$\rightarrow$

(b) $\frac{dy}{dt} + 4y(t) = \delta(t)$

For $t>0$ $\quad \frac{dy}{dt} + 4y(t) = 0 \Rightarrow y(t) = Ae^{-4t}$

J.C. $\quad \int_{0^-}^{0^+} \frac{dy}{dt}\,dt + \int_{0^-}^{0^+} 4y(t)\,dt = \int_{0^-}^{0^+} \delta(t)\,dt$

$\therefore y(0+) - \cancel{y(0-)}^{\,0} + \overset{0}{0} = 1 \qquad \therefore y(0+) = 1$

$\therefore$ applying J.C. $\quad y(0+) = 1 = Ae^{0} \Rightarrow A = 1$

$\therefore \boxed{h(t) = e^{-4t}u(t)}$

(c) $y(t) = h(t) * x(t) = e^{-4t}u(t) * e^{-t}u(t)$

$= \int_{-\infty}^{+\infty} e^{-4\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)\,d\tau = \int_{0}^{t} e^{-3\tau}\,d\tau\, e^{-t} \quad \underline{t \geq 0}$

$= e^{-t}\left[\frac{1}{-3}(e^{-3\tau})\Big|_0^t\right] = \frac{e^{-t}}{-3}(e^{-3t} - 1)$

$$\boxed{y(t) = \left(\frac{e^{-4t}}{-3} + \frac{1}{3}e^{-t}\right)u(t)}$$

3. (Difference equations)

(a) Consider the discrete LTI system represented by

$$y[n] = x[n] - x[n-1]$$

where the system is initially at rest. Determine impulse response $h[n]$. Plot it. Is the system stable?

(b) Now consider the discrete LTI system given by

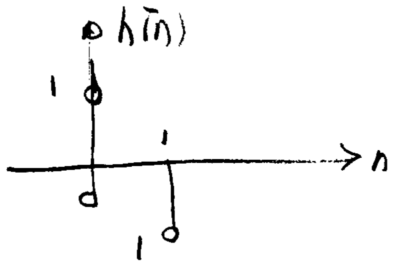
$$y[n] = y[n-1] + x[n]$$

where the system is initially at rest. Determine the impulse response $h[n]$. Plot it. Is the system stable?

(c) For the system in (b), determine step response $y[n]$ when $x[n] = u[n]$. Plot it.

(a) $y[n) = x(n) - x[n-1]$ $\qquad x[n) = \delta[n)$

$\therefore h[n) = \delta[n) - \delta[n-1)$



$\sum_{n=-\infty}^{+\infty} |h(n)| = |h(0)| + |h(1)|$

$= 2 < \infty$ $\boxed{\therefore \text{ STABLE}}$

(b) $y[n) = y[n-1) + x[n)$ $\qquad x[n) = \delta[n)$

for $n > 0$ $\quad y[n) - y[n-1) = 0$

Try $y[n) = A\alpha^n$ $\quad \therefore A\alpha^n - A\alpha^{n-1} \stackrel{?}{=} 0$

i.e. $A\alpha^n[1 - \alpha^{-1}] \stackrel{?}{=} 0$ Yes, if $\alpha = 1$

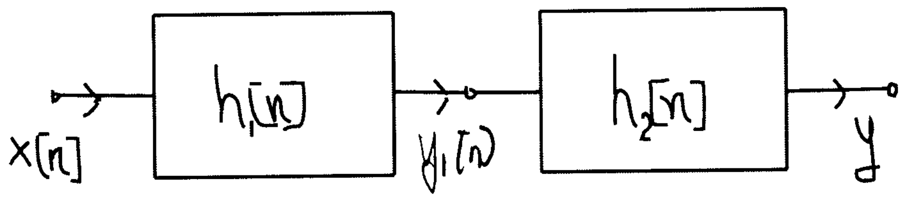
$\therefore y[n) \; [h(n)] = A 1^n \quad n > 0$

$y[0) = y[-1) + 1$ $\quad \therefore y(0) = 1$. Using that, get $\boxed{h[n) = 1 \quad n \geq 0}$ $u[n]$

$\sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = \infty$ $\quad \boxed{\therefore \text{UNSTABLE}}$

(c) $\boxed{y[n)} = u[n) * u[n) = \sum u[m) u[n-m) = \sum_{m=0}^{n} 1 = \boxed{(n+1) u[n)}$

4. (Discrete-time convolution)



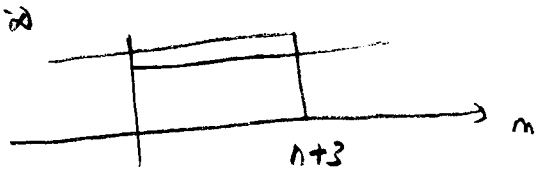
Consider the cascade of two discrete-time LTI systems where,

$$x[n] = (0.5)^n u[n]$$
$$h_1[n] = u[n+3]$$
$$h_2[n] = \delta[n] - \delta[n-1]$$

Determine output $y[n]$.

(i) $$y_1[n] = \sum_{m=-\infty}^{+\infty} (0.5)^m u[m]\, u[n-m+3] = \sum_{m=0}^{n+3} (0.5)^m = \frac{1-(0.5)^{n+4}}{1-(0.5)}$$

$n+3 \geq 0$

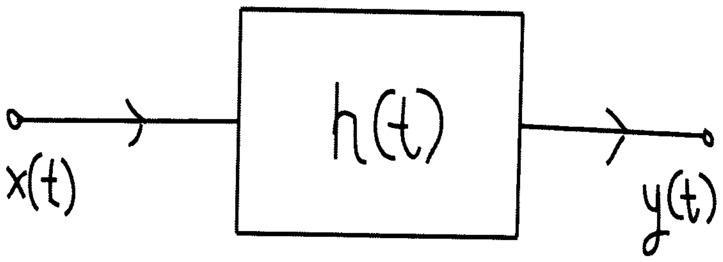


$$\therefore y_1[n] = \left(2 - 2(0.5)^{n+4}\right) u[n-3]$$

(ii) $$y(n) = y_1(n) * \left[\delta[n] - \delta[n-1]\right]$$
$$= y_1(n) - y_1(n-1)$$

$$\boxed{y[n] = \left[2 - 2(0.5)^{n+4}\right] u[n-3] - \left[2 - 2(0.5)^{n+3}\right] u[n-4]}$$

5.(Continuous-time convolution)



Let the input to a LTI system be $x(t) = e^{-2t}u(t)$. Consider the situation when the impulse response has a "natural frequency" which is the same as that of the excitation. That is, $h(t) = e^{-2t}u(t)$. Do the following:

(a) What is the natural frequency of the system?

(b) Determine $y(t)$ using continuous-time domain convolution.

(a) $h(t) = e^{-2t}u(t) \qquad e^{st}u(t)\Big|_{s=-2} \qquad \boxed{s=-2}$

(b) $$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} e^{-2\tau}u(\tau)\,e^{-2(t-\tau)}u(t-\tau)\,d\tau$$

$$= \int_0^t e^{-2\tau}e^{+2\tau}\,d\tau\, e^{-2t} \qquad t \geq 0$$

$$\boxed{y(t)} = \int_0^t 1\,d\tau\, e^{-2t} = \boxed{te^{-2t}u(t)}$$

6. (Discrete-time Fourier series, frequency response and output)

Consider a LTI digital filter with impulse response $h[n]$ and frequency response $H(e^{j\omega})$ where

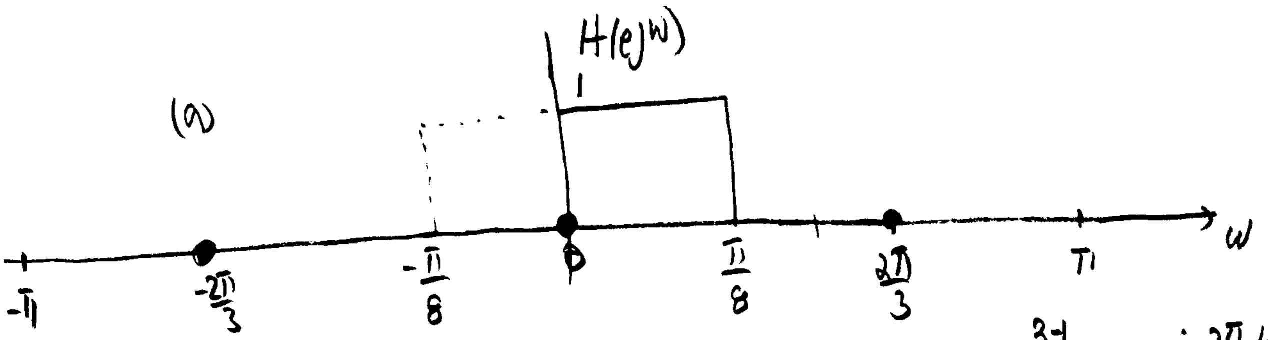
$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| < \pi \end{cases}$$

(a) Sketch $H(e^{j\omega})$ for $0 \le \omega \le \pi$.

(b) Consider an input signal $x[n]$ to the digital filter, where $x[n]$ is a periodic signal with period 3. It will have three Fourier coefficients. What are the three *frequencies* associated with these three coefficients? (Note: You do not need to find the Fourier coefficients. Just the frequencies at which they would occur in a 3-point discrete-time Fourier series).

(c) Show that the output $y[n]$ will have only one Fourier series coefficient per period.

(a)



(b) $x[n] = x[n+3]$ $\quad \underline{N=3}$ $\quad \therefore\ x[n] = \sum_{n=0}^{3-1} \hat{a}_k e^{j\frac{2\pi}{3}kn}$

$\therefore\ x[n] = \hat{a}_0 e^{0} + \hat{a}_1 e^{j\frac{2\pi}{3}1n} + \hat{a}_2 e^{j\frac{2\pi}{3}2n}$

3 Fourier coeffs. at frequencies $\boxed{0, \frac{2\pi}{3}, \frac{4\pi}{3}}$

Equivalently $0, \frac{2\pi}{3}, -\frac{2\pi}{3}$

(c) only the dc component gets through.

7. (Continuous-time Fourier series)

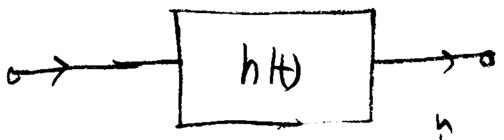
Consider the causal continuous-time LTI system of problem (2) whose input $x(t)$ and output $y(t)$ are related by the following differential equation,

$$\frac{d}{dt}y(t) + 4y(t) = x(t).$$

Find the output $y(t)$ when the input $x(t) = Cos 2\pi t$, $-\infty < t < +\infty$. (Note that you can do this, assuming you have a periodic input. Then the output will also be periodic).

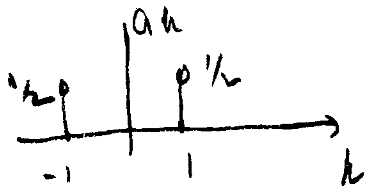
System $\frac{dy}{dt} + 4y(t) = x(t)$

$h(t) = e^{-4t}u(t)$ $\quad \therefore \boxed{H(j\Omega) = \frac{1}{j\Omega + 4}}$



$\hat{a}_k$ $\qquad \hat{b}_k = \hat{a}_k H(j\Omega)\big|_{\Omega = k\Omega_0}$

$x(t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$ $\qquad \Omega_0 = 2\pi \quad T_0 = 1$



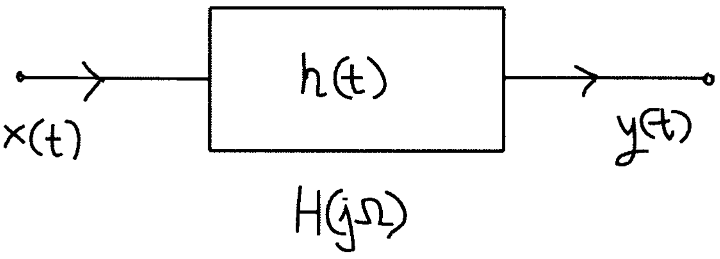
$\hat{b}_k = \hat{a}_k \frac{1}{(jk\Omega_0 + 4)}$

$$y(t) = \sum_{k=-1,1} \hat{b}_k e^{jk2\pi t} = \frac{\frac{1}{2}}{j(-1)2\pi + 4} e^{j(-1)2\pi t} + \frac{\frac{1}{2}}{j(1)2\pi + 4} e^{j(1)2\pi t}$$

$$= \frac{1/2}{4 + j2\pi} e^{j2\pi t} + \frac{1/2}{4 - j\pi} e^{-j2\pi t} = 2\,\mathrm{Re}\left\{\frac{1}{2}\frac{1}{\sqrt{16+4\pi^2}\,e^{j\theta}} e^{j2\pi t}\right\}$$

$$\boxed{y(t) = \frac{1}{\sqrt{16+4\pi^2}} \cos(2\pi t - \theta)} \qquad \theta = \tan^{-1}\pi/2$$

8. (Fourier transform)



Consider a causal LTI systems with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}$$

For a particular signal $x(t)$, the output is observed to be

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

(a)Determine $x(t)$.

(b) Give the differential equation describing the output $y(t)$ and input $x(t)$.

(a) $$Y(j\Omega) = \frac{1}{j\Omega+3} - \frac{1}{j\Omega+4} = \frac{1}{(j\Omega+3)(j\Omega+4)} = H(j\Omega)X(j\Omega)$$

$$X(j\Omega) = \frac{1}{(j\Omega+3)(j\Omega+4)\cdot H(j\Omega)} = \frac{1}{j\Omega+4}$$

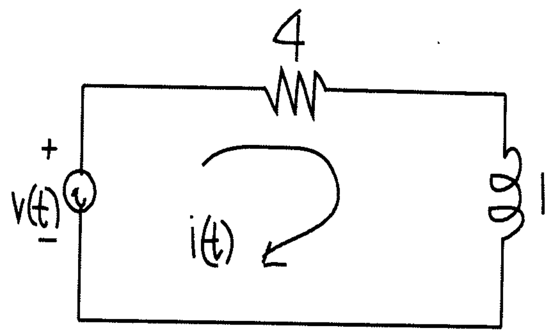
$$\boxed{\therefore x(t) = e^{-4t}u(t)}$$

(b) $$\frac{Y(j\Omega)}{X(j\Omega)} \overset{\circ}{=} H(j\Omega) = \frac{1}{j\Omega+3}$$

$$\therefore (j\Omega+3)Y(j\Omega) = X(j\Omega)$$

$$\boxed{\therefore \frac{dy}{dt} + 3y(t) = x(t)}$$

9. (Laplace transform)



Assume that a voltage $v(t) = e^{-3t}u(t)$ is applied at $t = 0$. Using Laplace transforms, find output current $i(t)$ for $t \geq 0$. Assume all initial conditions are zero.

$$1\frac{di(t)}{dt} + 4i(t) = v(t) = e^{-3t}u(t)$$

$$sI(s) - \cancel{i(0^-)}^{\,0} + 4I(s) = \frac{1}{s+3}$$

$$I(s)[s+4] = \frac{1}{s+3} \qquad I(s) = \frac{1}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$= \frac{1}{s+3} - \frac{1}{s+4}$$

$$\therefore \boxed{i(t) = e^{-3t}u(t) - e^{-4t}u(t)}$$

10. (Laplace Transform)

For the following transfer functions $H(s)$, show the poles on the s-plane for each transfer function. Determine the corresponding time-domain response.

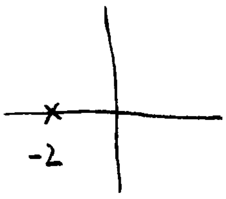
$H(s) = \frac{1}{s+2}$, $\quad Re\{s\} > -2$

$H(s) = \frac{1}{s-2}$, $\quad Re\{s\} > 2$

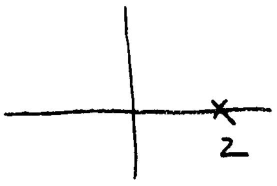
$H(s) = \frac{1}{(s+2)^2}$, $\quad Re\{s\} > -2$

$H(s) = \frac{1}{s+j2}$, $\quad Re\{s\} > 0$

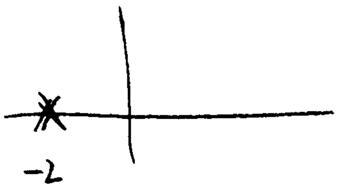
$H(s) = \frac{1}{s^2+4}$, $\quad Re\{s\} > 0$



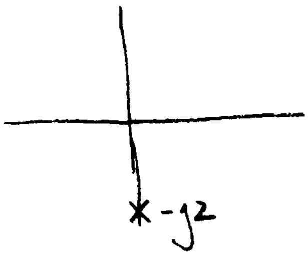
$\boxed{e^{-2t}u(t)}$



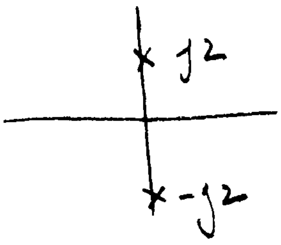
$\boxed{e^{2t}u(t)}$



$\boxed{te^{-2t}u(t)}$



$\boxed{e^{-2jt}u(t)}$



$e^{j2t}u(t) + e^{-jt}u(t) = \boxed{2\cos t\, u(t)}$