A Frequency Domain Method for Generation of Discrete-Time Analytic Signals

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Abstract

We consider a common frequency domain procedure *hilbert* for generating discrete-time analytic signals and show how it fails for a specific class of signals. A new frequency domain technique *ehilbert* is formulated that solves the defect. Moreover, the new technique is applicable to all discrete-time real signals of even length. It is implemented by the introduction of one additional zero of the continuous spectrum of the analytic signal *hilbert* at a negative frequency. Both frequency-domain methods have the same redundancy. The new analytic signal preserves the original signal (real part) and also the zeros of the discrete spectrum *hilbert* in the negative frequencies. The greater attenuation at the negative frequencies affects the degree of aliasing of the analytic signal. It is measured by applying the analytic signal to an orthogonal wavelet transform and determining the improved transform shiftability.

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I. INTRODUCTION

A continuous-time analytic signal is a complex time function having a Fourier transform equal to zero for all $\omega < 0$ ([5], Sec.1.11). For a complex time sequence, we cannot require the same constraint since the discrete-time Fourier transform (DTFT) spectrum is periodic. Instead, a complex sequence is defined to be "analytic" [8] by requiring its DTFT vanish in the interval $[-\pi, 0)$. Realizable approximations to this ideal characteristic are obtained in a number of ways. We will refer to these realizations also as discrete-time analytic (DTA) signals or when the context is clear, as analytic signals. One motivation for transforming a real valued signal to an analytic one stems from the conjugate symmetric property of the DTFT of real signals. This allows a real signal to be recovered from its complex counterpart. That is, the spectral information contained in $[-\pi, 0)$ is redundant. Hence, removing the spectra at those frequencies, while preserving that at positive components, will not affect the information contained in the resulting signal.

The advantage of processing DTA signals is seen in many applications: as a first example, the one-sided spectrum preserves bandwidth resources; hence these signals find use in the design of single-sideband convertors [6]. Other examples include discrete wavelet transforms, where we see a reduction of shift sensitivity and improved directionality [2] and spectral analysis [1], where they are used in the estimation of instantaneous frequency. Methods currently used to generate DTA signals are either time domain [10], [3], [8] filtering methods or frequency-based ones [7], [5]. The former requires the generation of filter coefficients; the analytic signal is then obtained as the output of these filters. Filter coefficients can be obtained by windowing or equiripple methods, and are based on approximations to the continuous spectrum of either the ideal Hilbert transformer or the ideal lowpass filter. The second approach consists of setting the spectrum at the negative frequencies to zero via the discrete Fourier transform (DFT), and subsequent generation of the DTA through an inverse DFT. In the time domain approach, the length of the filter can affect the accuracy of the approximation to the analytic signal. For instance in [10], the number of taps needed is large (128), but this makes the filter inappropriate for small-length signals. In the frequency domain approach, we will observe that the method fails for a specific class of signals, in that analytic signals are not generated. In both cases, in application in wavelet decomposition, the issue of reduced aliasing or equivalently, transform shiftability [9] needs to be considered.

We describe here a new method for generating a DTA signal [4]. This procedure, which is also a frequency domain approach, resolves the problem stated earlier. That is, where the frequency-domain method fails for a certain class of signals, the new method is successful. The new method has general applicability and is seen to improve transform shiftability. A specific formulation for the DTA signal, using the standard method [7], [5], is derived in Section II. Using that expression, we show how the method fails for a certain class of signals. In Section III, that same formulation is used to develop a new DTA signal based on its DTFT. Furthermore, the new method is applicable to all discrete-time real signals of even length for generation of DTA signals. In addition, we see that these DTA signals lead to improved transform shiftability. The new method is seen to be an extension of the standard frequency domain approach. In Section IV, time and frequency domain methods are compared for the degree of aliasing generated, as measured by the determination of shiftability. Conclusions are presented in Section V.

II. DISCRETE ANALYTIC SIGNAL VIA DFT

The concept of a DTA signal was formulated in [8]: For $x(n)$ a finite real valued sequence, the DTA signal $z(n)$ is defined as

$$z(n) = x(n) + jH\{x(n)\}$$

where H is the Hilbert transform operator, and $j^2 = -1$. The periodic spectrum of $z(n)$ is

$$Z(e^{j\omega}) = \sum_{n=0}^{N-1} z(n)e^{-j\omega n}$$

where N is the length of $z(n)$. The DTFT is periodic with period 2π. Thus ω is considered in the interval $[-\pi, \pi]$. For analyticity it is required [[8], Sec.11.4] that $Z(e^{j\omega}) = 0$ for $\omega \in [-\pi, 0)$. Because ω is continuous in the interval $[-\pi, \pi]$, the DTFT cannot be computed exactly. Hence the necessity for using the DFT. The DFT is obtained by uniformly sampling the DTFT on the ω-axis at $\omega_k = 2\pi k/N$, where $0\leq k \leq N-1$. Thus

$$Z(k) = Z(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} z(n)e^{-j2\pi kn/N}.$$

A common approach to generating a DTA signal resides in the frequency domain [7], [5]. We assume N to be even (the case for N odd is easily handled (see [5], page 145)). The procedure for deriving the DTA signal consists of three steps:

- Compute the N-point DFT of $x(n)$.
- Form the N-point DFT of the corresponding DTA signal by multiplying the N-point DFT of $x(n)$ by the vector:

$$a(n) = \begin{cases} 1, & n = 0 \\ 2, & 1 \leq n \leq N/2 - 1 \\ 1, & n = N/2 \\ 0, & N/2 + 1 \leq n \leq N - 1. \end{cases}$$

Thus, the N-point DFT of the DTA signal is

$$Z(k) = \begin{cases} X(k), & k = 0 \\ 2X(k), & 1 \leq k \leq N/2 - 1 \\ X(k), & k = N/2 \\ 0, & N/2 + 1 \leq k \leq N - 1. \end{cases} \tag{1}$$

- Obtain the DTA signal by computing the inverse DFT of the N-point DFT :

$$z(n) = 1/N \sum_{k=0}^{N-1} Z(k)e^{j2\pi kn/N}. \tag{2}$$

For our purposes, we need to derive an explicit expression for $z(n)$ in terms of the real signal $x(n)$.

Proposition I: For even integer N, $z(n)$ can be written as

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p+1) \cot(\pi(n-(2p+1))/N), \qquad \text{if } n \text{ even} \tag{3}$$

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n-2p)/N), \qquad \text{if } n \text{ odd.} \tag{4}$$

Proof : See Appendix A.

As stated earlier, a continuous-time analytic signal is defined as that with spectrum which is zero for $\omega < 0$. Hence it is a complex signal. Thus a non-zero real signal is not analytic. Similarly, a non-zero discrete-time real signal is not analytic. From equations (3) and (4), we observe that were the imaginary part of $z(n)$ to be zero, then $z(n)$ would be real and consequently not analytic. Hence consider the following situation: Suppose all even values of $x(n)$ are equal to some real constant α and all odd values to some real constant β (with no loss of generality we assume both constants to be different from zero). Then we have from equations (3) and (4)

$$\text{imag}(z(n)) = (2/N)\beta \sum_{p=0}^{N/2-1} \cot(\pi(n-(2p+1))/N), \qquad \text{if } n \text{ even}$$

$$\text{imag}(z(n)) = (2/N)\alpha \sum_{p=0}^{N/2-1} \cot(\pi(n-2p)/N), \qquad \text{if } n \text{ odd.}$$

The imaginary part of $z(n)$ is zero is equivalent to

$$\sum_{p=0}^{N/2-1} \cot(\pi(n-(2p+1))/N) = 0, \qquad \text{if } n \text{ even} \tag{5}$$

and

$$\sum_{p=0}^{N/2-1} \cot(\pi(n-2p)/N) = 0, \qquad \text{if } n \text{ odd.} \tag{6}$$

Proposition II: For N even,

$$\sum_{p=0}^{N/2-1} \cot(\pi(n-(2p+1))/N) = 0, \qquad \text{if } n \text{ even} \tag{7}$$

and

$$\sum_{p=0}^{N/2-1} \cot(\pi(n-2p)/N) = 0, \qquad \text{if } n \text{ odd.} \tag{8}$$

Proof: See Appendix B.

Accordingly, we conclude that for this specific class of signals, $z(n)$ is real and hence a DTA signal is not generated. We note that this class of signals, where the analytic mapping reduces to the identity map, are defined by only two frequencies, 0 and π.

Example: Let $N = 4$. Let $x(n) = [x(0) \quad x(1) \quad x(2) \quad x(3)]$ be a discrete real valued signal. Thus from equations (3), (4) the DTA $z(n)$ corresponding to $x(n)$ is

$$z(0) = x(0) + \frac{j}{2}(x(3) - x(1))$$

$$z(1) = x(1) + \frac{j}{2}(x(0) - x(2))$$

$$z(2) = x(2) + \frac{j}{2}(x(1) - x(3))$$

$$z(3) = x(3) + \frac{j}{2}(x(2) - x(0)).$$

For $x(3) = x(1) = 1$ and $x(2) = x(0) = 2$ we get $z(n) = x(n)$. Also, using MATLAB6.5, we see that:

$$\mathit{hilbert}([1 \quad 2 \quad 1 \quad 2]) = [1 \quad 2 \quad 1 \quad 2];$$

Note that MATLAB6.5 uses the algorithm in [7] and the DTA signal is generated by the function *hilbert*.

III. THE NEW METHOD

We have seen an example where the algorithm in [7] fails to generate the corresponding DTA signal. The new method for resolving this problem builds on the DTA signal formulation of equations (3) and (4). We make a simple modification to the imaginary part while ensuring that the real part is unchanged. This becomes a frequency domain approach where the algorithm in [7] corresponds to no modification. We label the new method *ehilbert*. As we will observe, this procedure in the frequency domain results in the addition of an imaginary number to the DC and the Nyquist frequency terms of the DTA signal obtained using *hilbert.* This guarantees that the DTFT (and hence the DFT) equals zero at $\omega_k = 2\pi k/N$, $N/2+1 \leq k \leq N-1$. In addition, the value of the DTFT can be made zero at *another* negative frequency of our choice. This resolves the problem faced earlier where the analytic map reduced to an identity map. Furthermore, it also forces the continuous spectrum to be small around a region of a point in $(-\pi, 0)$. This forms a DTA signal with reduced bandwidth. When applied to an orthogonal filterbank, the corresponding reduction in aliasing leads to improved transform shiftability. In addition, we have ensured that the real part of the corresponding analytic signal is the original signal. We proceed as follows:

Let $x(n)$ be a finite real valued sequence of length N. Recall that N is even. We restrict ourselves to only this case. With reference to equations (3) and (4), let $s(n)$ be an analytic signal such that

$$s(n) \quad = \quad x(n) + j(2/N)\{\sum_{p=0}^{N/2-1} x(2p+1)\cot(\pi(n-(2p+1))/N) + a\}, \qquad \text{if } n \text{ even} \tag{9}$$

$$s(n) = x(n) + j(2/N)\{\sum_{p=0}^{N/2-1} x(2p)\cot(\pi(n-2p)/N) + b\}, \qquad \text{if } n \text{ odd} \tag{10}$$

where we have added real variables $\frac{2}{N}a$ and $\frac{2}{N}b$ to the imaginary parts of the analytic signal $z(n)$ of equations (3) and (4). We will show that the DFT $S(k)$ of $s(n)$ is zero for $N/2+1 \le k \le N-1$. Thus, for n even

$$s(n) = z(n) + j(2/N)a = z(n) + jt(n)$$

and for n odd

$$s(n) = z(n) + j(2/N)b = z(n) + jt(n)$$

where

$$t(n) = \begin{cases} 2a/N, & n \textit{ even} \\ 2b/N, & n \textit{ odd.} \end{cases}$$

The DFT of $t(n)$ is equal to

$$\begin{aligned} T(k) &= \sum_{n=0}^{N-1} t(n)e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N/2-1} t(2n)e^{-j2\pi k2n/N} + \sum_{n=0}^{N/2-1} t(2n+1)e^{-j2\pi k(2n+1)/N} \\ &= (2/N)\{a \sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} + b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}\}. \end{aligned}$$

Thus

$$\begin{aligned} S(k) &= Z(k) + jT(k) \\ &= Z(k) + j\{(2/N)\{a \sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} + b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}\}\} \end{aligned}$$

where $Z(k)$ is the DFT of $z(n)$. We now show that

$$S(k) = \begin{cases} X(k) + j(a+b), & k = 0 \\ 2X(k), & 1 \le k \le N/2-1 \\ X(k) + j(a-b), & k = N/2 \\ 0, & N/2+1 \le k \le N-1. \end{cases} \tag{11}$$

Recall that $Z(k)$ satisfies the following

$$Z(k) = \begin{cases} X(k), & k = 0 \\ 2X(k), & 1 \leq k \leq N/2 - 1 \\ X(k), & k = N/2 \\ 0, & N/2 + 1 \leq k \leq N - 1. \end{cases}$$

Therefore it is sufficient to show that

$$T(k) = \begin{cases} a + b, & k = 0 \\ 0, & 1 \leq k \leq N/2 - 1 \\ a - b, & k = N/2 \\ 0, & N/2 + 1 \leq k \leq N - 1. \end{cases} \tag{12}$$

The proofs for $k = 0$ and $k = N/2$ are straightforward. For $1 \leq k \leq N/2 - 1$ it is easily verified that $\sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} = \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N} = 0$. For the remaining interval the proof follows by the conjugate symmetry relationship for real signals. Having established equation (11), where $S(k)$ has a one sided spectrum like $Z(k)$, parameters a and b are available to establish other effects. We utilize this degree of freedom, that is, we choose values for a and b to force the DTFT of $s(n)$ to be zero for some ω in the interval $(-\pi, 0)$. This, as we shall see, alleviates the problem encountered before, by generating a complex signal rather than a real one for the special class of signals referred to earlier. Zeroing the DTFT for some ω in the negative frequency range will also form a neighborhood of ω where the DTFT could be small, thus allowing for improved transform shiftability. Hence, we need to determine a and b such that for some ω

$$S(e^{j\omega}) = \sum_{n=0}^{N-1} s(n)e^{-j\omega n} = 0 \tag{13}$$

where $s(n)$ is defined by equations (9) and (10). We first need to establish that a solution exists. That is shown as follows:

Proposition III:

For, $s(n)$ defined by equations (9) and (10), and $\omega \in (-\pi, 0)$, $\exists$ real numbers a, b such that

$$S(e^{j\omega}) = \sum_{n=0}^{N-1} s(n)e^{-j\omega n} = 0 \tag{14}$$

Proof: See Appendix C.

In practice, we proceed by selecting an ω in $(-\pi, 0)$ and determining the corresponding values of a and b using equation (21).

Example: The method described above was used to generate an analytic signal from the sequence $x(n) = [1 \quad 2 \quad 1 \quad 2]$. We also consider another signal $h(n)$, the daubechies scaling filter db16 not belonging to the previous class. We choose constants a and b such that the DTFT is equal to zero at $\omega = -2.4$ for $x(n)$ and $\omega = -\pi + 0.001$ for $h(n)$ of the filter. The ω values were selected empirically. Results are compared with the analytic signals obtained using *hilbert*. For both cases we illustrate the results using 256 point DFTs. For

$x(n) = [1 \quad 2 \quad 1 \quad 2]$, we get $z(n) = x(n)$ using *hilbert* and $s(n) = [1-1.869j \quad 2-0.702j \quad 1-1.869j \quad 2-0.702j]$ using *ehilbert*. Figure (1) shows the magnitude of the spectra of the analytic signals generated by both methods. As formulated, the spectrum using *ehilbert* is equal to zero at $\omega = -2.4$, and small in the neighborhood of $\omega = -2.4$. This is not the case when using *hilbert*. As expected, the magnitude of the spectrum for the latter case is *symmetric*. For the db16 signal, Figure (2) shows the spectrum by both methods. We note that the magnitude of the spectrum of the DTA signal generated by *ehilbert* vanishes faster than that for the spectrum generated by *hilbert*.

IV. EXPERIMENTAL RESULTS

The new method relies on the introduction of an additional zero of the continuous spectrum of the analytic signal given by *hilbert*, in the region $(-\pi, 0)$. This causes the spectrum to be small about that point, thereby potentially reducing the analytic signal bandwidth. We determine the degree of bandwidth reduction or equivalently, sideband suppression here, in terms of the reduction in aliasing. DTA signals generated by time and frequency domain methods are applied to an orthogonal wavelet transform. The corresponding reduction in aliasing is measured at the subbands.

To measure the reduction in aliasing we use the concept of shiftability [9]. Shiftability, in the spatial domain, corresponds to a lack of aliasing. It is equivalent to the constraint that the power of the transform coefficients is preserved when the input signal is shifted in position (ibid., Proposition 2, pg.591). Consequently, we determine the degree of reduction in aliasing by determining the variation in subband power as the input analytic signal is translated.

As described earlier, methods for generation of DTA signals exist in the time and frequency domains. These methods do have some fundamental differences. Time-domain methods [10], [3], [8] require the generation of filter coefficients, the analytic signal being obtained as the output of these filters. They are based on approximations to a continuous spectrum. The length of the corresponding DTA signal depends on both the signal and filter lengths. Frequency domain methods *hilbert, ehilbert* on the other hand are not real-time, requiring access to the whole signal. The corresponding DTA signal is obtained from the discrete spectrum of the real signal. The length of the DTA signal depends only on the length of the original real signal.

For our experiment to be consistent with all four methods, we generated an "analytical impulse", that is, a DTA signal corresponding to an impulse. Such DTA signals of length 16 were generated using *hilbert*, *ehilbert*, *projection* [2] and *remez* [10] algorithms. The latter two methods entail approximation of lowpass filters, which are then shifted by $\pi/2$. A daubechies filter is used in the *projection* method while *remez* uses the equiripple approximation. Each of the DTA signals was padded with 16 zeros and applied to a discrete orthogonal wavelet filterbank using Meyer's scaling and wavelet filters. Subband powers at three scales were plotted over 16 shifts of the input signal, the input signal power being constant over the shifts. Figures (3) and (4) show the transform powers at the four subbands, as a function of input signal shifts.

We observe that power at all four subbands using *hilbert*, varies substantially over input signal shifts, relative to that with *ehilbert*. At level 2, performance of *hilbert*, *projection* and *remez* are about equal. At subbands at level 3 *ehilbert* performs considerably better than *projection* for which the power varies substantially while power variations are about the same relative to *remez*.

V. CONCLUSION

We have proposed a new method *ehilbert* for generating a DTA signal for which the algorithm *hilbert* in Matlab6.5 is obtained by setting $(a = b = 0)$. The advantage of the method is that it assures better suppression of negative frequencies. The cost is a loss of orthogonality. Beside zeroing the DTFT of the DTA signal at $\omega_k = 2\pi k/N$, where $N/2 + 1 \leq k \leq N - 1$, it also zeros the DTFT of the DTA signal at a point in the negative frequency range, thus leading to reduced aliasing and hence improved shiftability. Determination of the optimal point remains an open problem. Frequency and time domain methods for generating DTA signals were compared using shiftability.

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APPENDIX A
PROOF OF PROPOSITION I

From equation (2) and using equation (1) we have

$$\begin{aligned}
z(n) &= 1/N \sum_{k=0}^{N/2} Z(k) e^{j2\pi kn/N} \\
&= \frac{1}{N} Z(0) + 1/N \sum_{k=1}^{N/2-1} Z(k) e^{j2\pi kn/N} + \frac{1}{N} Z(N/2) e^{j2\pi (N/2)n/N} \\
&= \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} + \frac{1}{N} X(N/2) e^{j\pi n} \\
&= \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} + \frac{1}{N} X(N/2) e^{j\pi n} + x(n) - x(n) \\
&= x(n) + \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} + \frac{1}{N} X(N/2) e^{j\pi n} - 1/N \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \\
&= x(n) + \frac{1}{N} X(0) + 2/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} - 1/N \sum_{k=0}^{N/2-1} X(k) e^{j2\pi kn/N} - 1/N \sum_{k=N/2+1}^{N-1} X(k) e^{j2\pi kn/N} \\
&= x(n) + 1/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} - 1/N \sum_{k=N/2+1}^{N-1} X(k) e^{j2\pi kn/N}. \qquad (15)
\end{aligned}$$

Let $B \equiv -1/N \sum_{k=N/2+1}^{N-1} X(k) e^{j2\pi kn/N}$ and change variable $k = p + \frac{N}{2}$. Thus

$$B = -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2}) e^{j2\pi (p+\frac{N}{2})n/N} = -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2}) e^{j2\pi pn/N} e^{j\pi n}$$

$$= -1/N \sum_{p=1}^{N/2-1} X(p + \frac{N}{2}) e^{j2\pi pn/N} (-1)^n.$$

We therefore have from equation (15)

$$\begin{aligned} z(n) &= x(n) + 1/N \sum_{k=1}^{N/2-1} X(k) e^{j2\pi kn/N} - 1/N \sum_{k=1}^{N/2-1} X(k + \frac{N}{2}) e^{j2\pi kn/N} (-1)^n \\ &= x(n) + 1/N \sum_{k=1}^{N/2-1} \{X(k) e^{j2\pi kn/N} - X(k + \frac{N}{2}) e^{j2\pi kn/N} (-1)^n\} \\ &= x(n) + 1/N \sum_{k=1}^{N/2-1} e^{j2\pi kn/N} \{X(k) - X(k + \frac{N}{2})(-1)^n\}. \end{aligned} \tag{16}$$

The terms in parenthesis in equation (16) is

$$\begin{aligned} X(k) - X(k + \frac{N}{2})(-1)^{-n} &= \sum_{m=0}^{N-1} x(m) e^{-j2\pi mk/N} - \sum_{m=0}^{N-1} x(m) e^{-j2\pi (k+N/2)m/N} (-1)^{-n} \\ &= \sum_{m=0}^{N-1} x(m) \{e^{-j2\pi mk/N} - e^{-j2\pi (k+N/2)m/N} (-1)^{-n}\} \\ &= \sum_{m=0}^{N-1} x(m) e^{-j2\pi mk/N} \{1 - (-1)^{m+n}\}. \end{aligned}$$

Accordingly, equation (16) is written as

$$\begin{aligned} z(n) &= x(n) + 1/N \sum_{k=1}^{N/2-1} e^{j2\pi kn/N} \{\sum_{m=0}^{N-1} x(m) e^{-j2\pi mk/N} \{1 - (-1)^{m+n}\}\} \\ &= x(n) + 1/N \{\sum_{m=0}^{N-1} x(m)(1 - (-1)^{m+n}) \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N}\}. \end{aligned} \tag{17}$$

Observe that in (17) if $(m+n)$ is even the term $x(m)(1 - (-1)^{m+n}) \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} = 0$; therefore we are interested in the case when is $(m+n)$ odd . Suppose $(m+n)$ is odd. We first simplify the expression $\sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N}$ in equation (17) as

$$\sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} = \sum_{k=0}^{N/2-1} e^{j2\pi k(n-m)/N} - 1 = \frac{1 - e^{j\pi(n-m)}}{1 - e^{j2\pi(n-m)/N}} - 1 = \frac{1 - (-1)^{(n-m)}}{1 - e^{j2\pi(n-m)/N}} - 1.$$

Now $(m+n)$ odd is equivalent to $(n-m)$ odd. Therefore

$$\begin{aligned} \sum_{k=1}^{N/2-1} e^{j2\pi k(n-m)/N} &= \frac{2}{1 - e^{j2\pi(n-m)/N}} - 1 = \frac{1 + e^{j2\pi(n-m)/N}}{1 - e^{j2\pi(n-m)/N}} = \frac{e^{-j\pi(n-m)/N} + e^{j\pi(n-m)/N}}{e^{-j\pi(n-m)/N} - e^{j\pi(n-m)/N}} \\ &= \frac{2\cos(\frac{\pi}{N}(n-m))}{-j2\sin(\frac{\pi}{N}(n-m)} = j\frac{\cos(\frac{\pi}{N}(n-m))}{\sin(\frac{\pi}{N}(n-m)} = j\cot(\frac{\pi}{N}(n-m)). \end{aligned} \tag{18}$$

For n even, $(n-m)$ odd is equivalent to m odd. Also for n odd, $(n-m)$ odd is equivalent to m even. Therefore, for n even and m odd, we have $1-(-1)^{n+m}=2$, and for n even and m even, we have $1-(-1)^{n+m}=0$. Thus, referring back to equation (17) and using (18), we conclude that for n even

$$\begin{aligned} z(n) &= x(n) + 1/N \sum_{m=0}^{N-1} x(m)(1-(-1)^{m+n}) j \cot(\frac{\pi}{N}(n-m)) \\ &= x(n) + 1/N \sum_{m=0,\, m \text{ odd}}^{N-1} x(m)(1-(-1)^{m+n}) j \cot(\frac{\pi}{N}(n-m)) \\ &= x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p+1) \cot(\pi(n-(2p+1))/N). \end{aligned}$$

Similarly for n odd, we can show that

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) \cot(\pi(n-2p)/N).$$

This concludes the proof.

APPENDIX B
PROOF OF PROPOSITION II

We can show that the conditions in equations (7), (8) are always true. The proof is in two steps for each equation. It is first shown that the LHS of each equation can be written as a circulant matrix; then, the first row of this matrix sums to zero. Equation (7) is proven first. Denote by $A = (a)_{\frac{n}{2}p}$ the matrix whose elements are $(a)_{\frac{n}{2}p} = \cot(\pi(n-(2p+1))/N)$, where $0 \le p \le \frac{N}{2}-1$, $0 \le n \le N-1$, and n even. Hence

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdot & \cdot & \cdot & \cdot & a_{0(\frac{N}{2}-1)} \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2(\frac{N}{2}-1)} \\ \cdot & a_{41} & a_{42} & a_{43} & a_{44} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{(N-6)(\frac{N}{2}-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{(N-4)(\frac{N}{2}-1)} \\ a_{(N-2)0} & a_{(N-2)1} & \cdot & \cdot & \cdot & \cdot & a_{(N-2)(\frac{N}{2}-2)} & a_{(N-2)(\frac{N}{2}-1)} \end{bmatrix}. \tag{19}$$

We need to show that the matrix $A = (a)_{\frac{n}{2}p}$, for $0 \le p \le \frac{N}{2}-1$, $0 \le n \le N-1$ and n even is a circulant matrix. This is equivalent to showing that it is Toeplitz and the first column is obtained by transposing the first row and flipping the last $(\frac{N}{2}-1)$ terms. We first show the Toeplitz property. Equivalently, we need to show that

$$a_{(\frac{n}{2}+1)(p+1)} = a_{\frac{n}{2}p}.$$

For $0 \leq p \leq \frac{N}{2}-2$, $0 \leq n < N-2$, n even, we have

$$\begin{aligned} a_{(\frac{n}{2}+1)(p+1)} &= \cot(\frac{\pi}{N}(n+2-2p-3)) \\ &= \cot(\frac{\pi}{N}(n-2p-1)) \\ &= a_{\frac{n}{2}p}. \end{aligned}$$

Hence A is Toeplitz. Next, we need to verify that given the first row, the first column is obtained by transposing the first row and flipping the last $(\frac{N}{2}-1)$ terms. This is equivalent to verifying that $a_{p0} = a_{0(\frac{N}{2}-p)}$ where $0 < p \leq \frac{N}{2}-1$. We see that

$$\begin{aligned} a_{0(\frac{N}{2}-p)} &= \cot(\frac{\pi}{N}(-(2(\frac{N}{2}-p)+1))) \\ &= \cot(\frac{\pi}{N}(2p-1)-\pi) \\ &= \cot(\frac{\pi}{N}(2p-1)) \\ &= a_{p0}. \end{aligned}$$

Therefore we can conclude that A is circulant. Now, in order to show $\sum_{p=0}^{N/2-1}\cot(\pi(n-(2p+1))/N) = 0$, $0 \leq n \leq (N-1)$ and n even, it is enough to show that the first row of the matrix A sums to zero. Then, for $n=0$ we get

$$\sum_{p=0}^{N/2-1}\cot(\pi(-(2p+1))/N) = \sum_{p=0}^{N/4-1}\cot(\pi(-(2p+1))/N) + \sum_{p=N/4}^{N/2-1}\cot(\pi(-(2p+1))/N).$$

Now

$$\begin{aligned} \sum_{p=N/4}^{N/2-1}\cot(\pi(-(2p+1))/N) &= \sum_{q=0}^{N/4-1}\cot(\frac{\pi}{N}(-(2(\frac{N}{2}-1-q)+1))) \\ &= \sum_{q=0}^{N/4-1}\cot(\frac{\pi}{N}(2q+1)-\pi) = \sum_{q=0}^{N/4-1}\cot(\frac{\pi}{N}(2q+1)). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{p=0}^{N/2-1}\cot(\pi(-(2p+1))/N) &= \sum_{p=0}^{N/4-1}\cot(\pi(-(2p+1))/N) + \sum_{q=0}^{N/4-1}\cot(\frac{\pi}{N}(2q+1)) \\ &= 0. \end{aligned}$$

Hence, for $0 \leq n \leq (N-1)$ and n even we have $\sum_{p=0}^{N/2-1}\cot(\pi(-(2p+1))/N) = 0$. Likewise, we can show that $\sum_{p=0}^{N/2-1}\cot(\pi(n-2p)/N) = 0$, $\forall\ 0 \leq n \leq (N-1)$ and n odd.

APPENDIX C
PROOF OF PROPOSITION III

We have

$$\begin{aligned} S(e^{j\omega}) &= \sum_{p=0}^{N-1} s(p)e^{-j\omega p} \\ &= \sum_{p=0}^{\frac{N}{2}-1} s(2p)e^{-j\omega 2p} + \sum_{p=1}^{\frac{N}{2}} s(2p-1)e^{-j\omega(2p-1)}. \end{aligned}$$

Denote by

$\alpha_1(\omega) = \sum_{p=0}^{N/2-1} x(2p)\cos(2p\omega)$

$\alpha_2(\omega) = 2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1)\cot(f(p,q))\sin(2p\omega)$

$r_1(\omega) = 2/N \sum_{p=0}^{N/2-1} \sin(2p\omega)$

$\alpha_3(\omega) = \sum_{p=1}^{N/2} x(2p-1)\cos(\omega(2p-1))$

$\alpha_4(\omega) = 2/N \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q)\cot(f(p,q))\sin(\omega(2p-1))$

$r_2(\omega) = 2/N \sum_{p=1}^{N/2} \sin(\omega(2p-1))$

$\alpha_{i1}(\omega) = -\sum_{p=0}^{N/2-1} x(2p)\sin(2p\omega)$

$\alpha_{i2}(\omega) = -2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1)\cot(f(p,q))\cos(2p\omega)$

$r_{i1}(\omega) = 2/N \sum_{p=0}^{N/2-1} \cos(2p\omega)$

$\alpha_{i3}(\omega) = -\sum_{p=1}^{N/2} x(2p-1)\sin(\omega(2p-1))$

$\alpha_{i4}(\omega) = -2/N \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q)\cot(f(p,q))\cos(\omega(2p-1))$

$r_{i2}(\omega) = 2/N \sum_{p=1}^{N/2} \cos(\omega(2p-1))$

where $f(p,q) = (\pi/N)(2p-(2q+1))$.

Therefore, using equation (9)

$$\begin{aligned} \sum_{p=0}^{\frac{N}{2}-1} s(2p)e^{-j\omega 2p} &= \sum_{p=0}^{\frac{N}{2}-1} \left\{ \left\{ x(2p) + j(2/N)\{ \sum_{q=0}^{N/2-1} x(2q+1)\cot(\pi(2p-(2q+1))/N) + a\} \right\} \right. \\ &\left. \left\{ \cos(2p\omega) - j\sin(2p\omega) \right\} \right\} \\ &= \sum_{p=0}^{\frac{N}{2}-1} x(2p)\cos(2p\omega) + 2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1)\cot(f(p,q))\sin(2p\omega) + \\ &a\frac{2}{N} \sum_{p=0}^{N/2-1} \sin(2p\omega) + \end{aligned}$$

$$j\Big\{-\sum_{p=0}^{N/2-1} x(2p)\sin(2p\omega) + 2/N \sum_{p=0}^{N/2-1}\sum_{q=0}^{N/2-1} x(2q+1)\cot(f(p,q))\cos(2p\omega)$$

$$+a\frac{2}{N}\sum_{p=0}^{N/2-1}\cos(2p\omega)\Big\}$$

$$= \alpha_1(\omega) + \alpha_2(\omega) + ar_1(\omega) + j\Big\{\alpha_{i1}(\omega) - \alpha_{i2}(\omega) + ar_{i1}(\omega)\Big\}$$

and using equation (10)

$$\begin{aligned}\sum_{p=1}^{\frac{N}{2}} s(2p-1)e^{-j\omega(2p-1)} &= \sum_{p=1}^{\frac{N}{2}}\Big\{\Big\{x(2p-1) + j(2/N)\{\sum_{q=0}^{N/2-1} x(2q)\cot(\pi(2p-1-2q)/N) + b\}\Big\} \\ &\Big\{\cos((2p-1)\omega) - j\sin((2p-1)\omega)\Big\}\Big\} \\ &= \sum_{p=1}^{N/2} x(2p-1)\cos(\omega(2p-1)) + 2/N\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}\Big\{x(2q)\cot(f(p,q))\sin(\omega(2p-1))\Big\} \\ &+b\frac{2}{N}\sum_{p=1}^{N/2}\sin(\omega(2p-1)) + j\Big\{-\sum_{p=1}^{N/2} x(2p-1)\sin(\omega(2p-1)) + \\ &\frac{2}{N}\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} x(2q)\cot(f(p,q))\cos(\omega(2p-1)) + b\frac{2}{N}\sum_{p=1}^{N/2}\cos(\omega(2p-1))\Big\} \\ &= \alpha_3(\omega) + \alpha_4(\omega) + br_2(\omega) + j\Big\{\alpha_{i3}(\omega) - \alpha_{i4}(\omega) + br_{i2}(\omega)\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}S(e^{j\omega}) &= \alpha_1(\omega) + \alpha_2(\omega) + ar_1(\omega) + \alpha_3(\omega) + \alpha_4(\omega) + br_2(\omega) + j\Big\{\alpha_{i1}(\omega) - \alpha_{i2}(\omega) + ar_{i1}(\omega) + \alpha_{i3}(\omega) - \\ &\alpha_{i4}(\omega) + br_{i2}(\omega)\Big\}\end{aligned} \tag{20}$$

To prove that a solution exists for $S(e^{j\omega}) = 0$, we proceed as follows: we equate the real and imaginary part of equation (20) to zero. Therefore

$$\begin{cases} ar_1(\omega) + br_2(\omega) = -\alpha_3(\omega) - \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega) \\ ar_{i1}(\omega) + br_{i2}(\omega) = -\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) + \alpha_{i2}(\omega). \end{cases}$$

Accordingly, for $\triangle(\omega) = -r_2(\omega)r_{i1}(\omega) + r_1(\omega)r_{i2}(\omega) \neq 0$, we have values for a and b as follows:

$$\begin{aligned} a &= \{-r_{i2}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) + \alpha_1(\omega) + \alpha_2(\omega)) - r_2(\omega)(-\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) + \alpha_{i2}(\omega))\}/\triangle(\omega) \\ b &= \{r_1(\omega)(-\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) + \alpha_{i2}(\omega)) + r_{i1}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) + \alpha_1(\omega) + \alpha_2(\omega))\}/\triangle(\omega). \end{aligned} \tag{21}$$

Now

$$
\begin{aligned}
\triangle(\omega) \quad = \quad & -(2/N)^2\{\sum_{p=1}^{N/2}\sin(\omega(2p-1))\sum_{q=0}^{N/2-1}\cos(2\omega q)-\sum_{p=1}^{N/2}\cos(\omega(2p-1))\sum_{q=0}^{N/2-1}\sin(2\omega q)\} \\
& = -(2/N)^2\{\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}\sin(\omega(2p-2q-1))\} \\
& = -(2/N)^2\{\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}\mathrm{imag}(e^{j(\omega(2p-2q-1))})\} \\
& = -(2/N)^2\{\mathrm{imag}(\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}e^{j(\omega(2p-2q-1))})\}.
\end{aligned}
$$

We can show that for $\omega \in (-\pi, 0)$

$$
imag(\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}e^{j(\omega(2p-2q-1))}) = \frac{1-\cos(\omega N)}{2\sin(\omega)}.
$$

We have

$$
\begin{aligned}
\sum_{p=1}^{\frac{N}{2}}\sum_{q=0}^{\frac{N}{2}-1}e^{j(\omega(2p-2q-1))} \quad = \quad & e^{-j\omega}\{\sum_{p=1}^{\frac{N}{2}}e^{j\omega 2p}\}\{\sum_{q=0}^{\frac{N}{2}-1}e^{-j\omega 2q}\} = e^{-j\omega}\{\sum_{p=1}^{\frac{N}{2}}(e^{j\omega 2})^p\}\{\sum_{q=0}^{\frac{N}{2}-1}(e^{-j\omega 2})^q\} \\
& = e^{-j\omega}\{\sum_{p=0}^{\frac{N}{2}}(e^{j\omega 2})^p-1\}\{\frac{1-e^{-j2\omega\frac{N}{2}}}{1-e^{-j2\omega}}\} = e^{-j\omega}\{\frac{1-e^{j2\omega(\frac{N}{2}+1)}}{1-e^{j2\omega}}-1\}\{\frac{1-e^{-j\omega N}}{1-e^{-j2\omega}}\} \\
& = e^{-j\omega}\frac{e^{j2\omega}-e^{j2\omega(\frac{N}{2}+1)}}{1-e^{j2\omega}}\frac{1-e^{-j\omega N}}{1-e^{-j2\omega}} = e^{-j\omega}e^{2j\omega}\frac{1-e^{j\omega N}}{1-e^{j2\omega}}\frac{1-e^{-j\omega N}}{1-e^{-j2\omega}} \\
& = e^{j\omega}\frac{1-e^{j\omega N}-e^{-j\omega N}+1}{1-e^{-j2\omega}-e^{j2\omega}+1} = e^{j\omega}\frac{1-\cos(\omega N)}{1-\cos(2\omega)} \\
& = e^{j\omega}\frac{1-\cos(\omega N)}{2\sin^2(\omega)} = (\cos(\omega)+j\sin(\omega))\frac{1-\cos(\omega N)}{2\sin^2(\omega)} \\
& = \cos(\omega)\frac{1-\cos(\omega N)}{2\sin^2(\omega)} + j\sin(\omega)\frac{1-\cos(\omega N)}{2\sin^2(\omega)}.
\end{aligned}
$$

Therefore

$$
imag(\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}e^{j(\omega(2p-2q-1))}) \quad = \quad \sin(\omega)\frac{1-\cos(\omega N)}{2\sin^2(\omega)} = \frac{1-\cos(\omega N)}{2\sin(\omega)}.
$$

For $\omega = -\pi$ we have

$$
\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1}e^{j(\omega(2p-2q-1))} = e^{j\pi}(\sum_{p=1}^{N/2}e^{-j(2\pi p)})(\sum_{q=0}^{N/2-1}e^{j(2\pi q)}) = -(\frac{N}{2})(\frac{N}{2}) = -(\frac{N}{2})^2.
$$

Therefore, for $\omega \in (-\pi, 0)$, $\triangle(\omega) = -(2/N)^2(1 - \cos(\omega N)/(2\sin(\omega)))$, and for $\omega = -\pi$, $\triangle(\omega) = 0$.

Thus, for $\omega \in (-\pi, 0)$ we have

$$\triangle(\omega) = 0 \leftrightarrow \cot(\omega N) = 1.$$

Hence

$$\triangle(\omega) = 0 \quad \leftrightarrow \quad \omega = 2\pi k/N \,\text{for}\, N/2 + 1 \leq k \leq N - 1.$$

Letting

$$A = \begin{cases} \omega \in [-\pi, 0)|\ \omega = 2\pi k/N \\ \text{where}\, N/2 + 1 \leq k \leq N - 1 \end{cases}$$

we conclude that for $\omega \in (-\pi, 0)$ and $\omega \notin A$ the system (21) has a solution. From equation (11), we observe that the DFT $S(k)$ of $s(n)$ is equal to zero on A. Therefore the DTFT, $S(e^{j\omega})$ is also zero on A.

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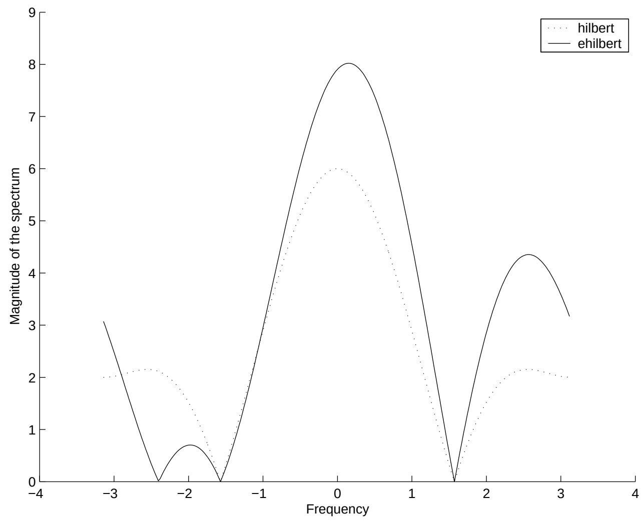

Figure 1: Spectrum of DTA signal for $x(n) = [1 \quad 2 \quad 1 \quad 2]$. For *ehilbert*, $S(e^{j\omega})$ forced to zero at $\omega = -2.4$.

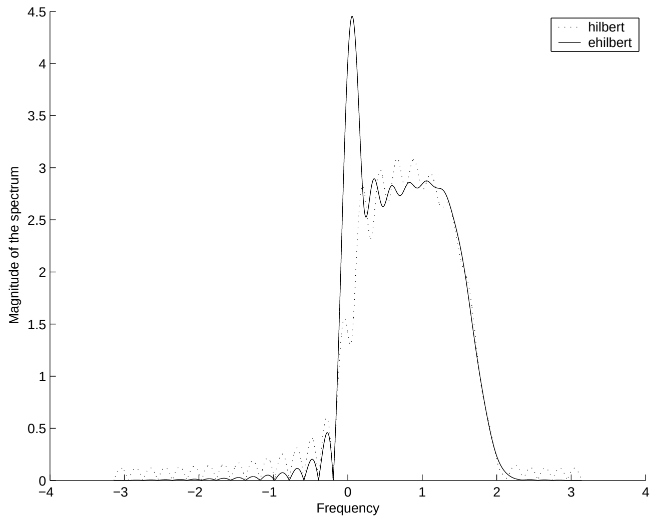

Figure 2: Spectrum of DTA signal for Daubechies scaling filter of length 32. For *ehilbert*, $S(e^{j\omega})$ forced to zero at $\omega = -\pi + 0.001$.

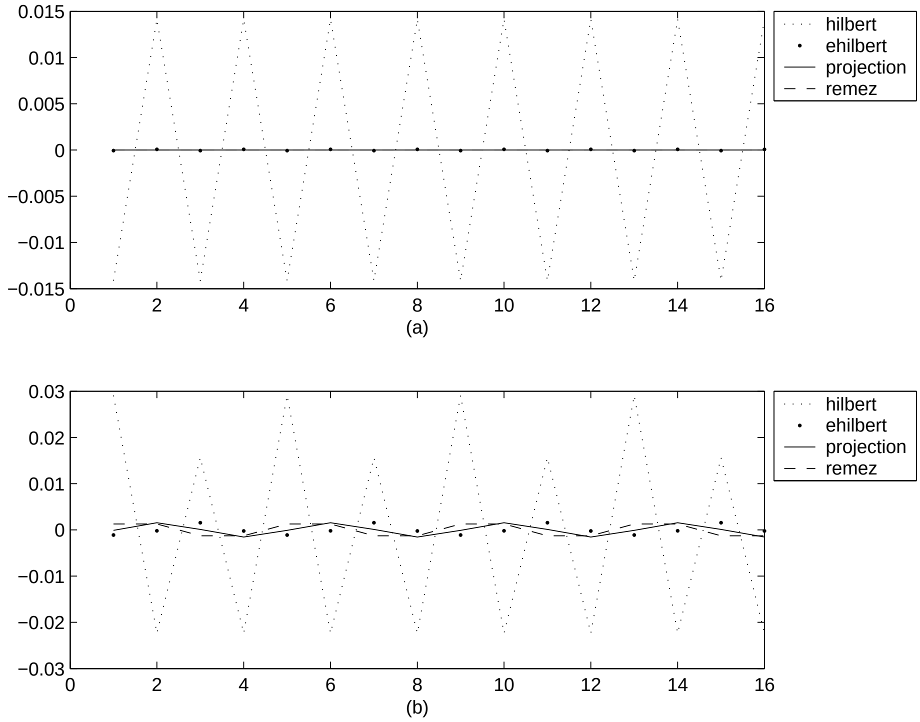

Figure 3: Subband power with four DTA signals. (a) Level-1 bandpass, (b) Level-2 bandpass. For *ehilbert*, $S(e^{j\omega})$ forced to zero at $\omega = -\pi + 2.31$

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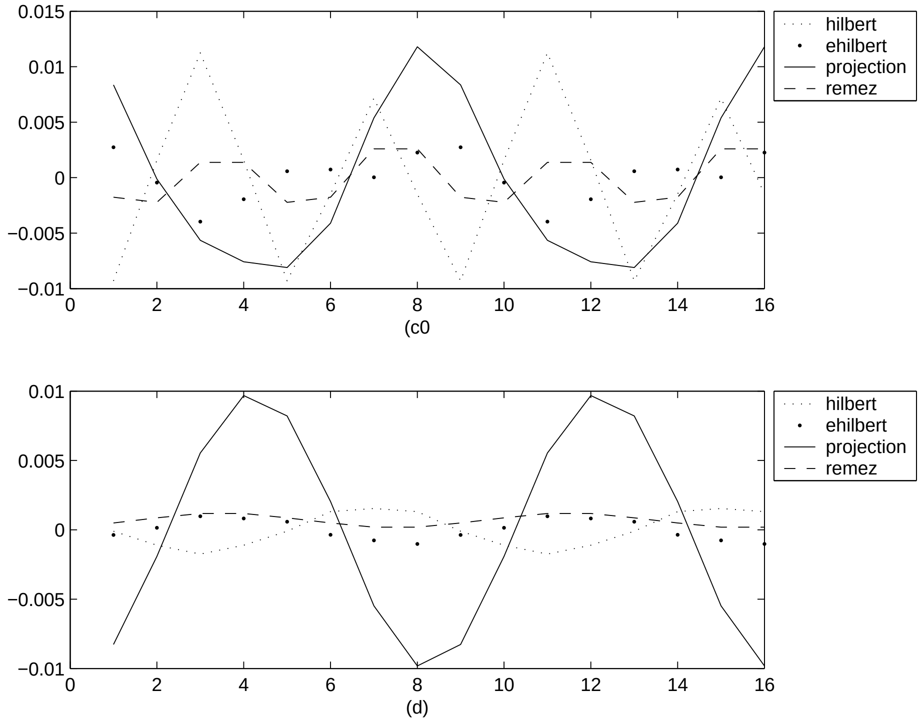

Figure 4: Subband power with four DTA signals. (c) Level-3 bandpass ,(d) Level-3 lowpass. For *ehilbert*, $S(e^{j\omega})$ forced to zero at $\omega = -\pi + 2.31$

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