A NOVEL METHOD FOR GENERATING COMPLEX HALF-BAND FILTERS

Mohamed Elfataoui and Gagan Mirchandani

Department of Electrical and Computer Engineering
The University of Vermont
Burlington, Vermont 05405, USA

ABSTRACT

We propose a simple, novel and efficient method for generating complex half-band FIR filters, which we use in the generation of discrete-time analytic (DTA) signals. These filters have properties of linear phase and real-time implementation while the DTA signals generated are orthogonal and invertible (the original real signal is recoverable). The filter design, in contrast to some other methods, is easily scalable and stable. The new method is evaluated for performance (i.e. aliasing) by comparing its shiftability [10] property with that of other transforms. Using a total variation measure for determining function variation, we see that its shiftability either matches or exceeds that of other methods. Furthermore, this design method lends itself to an enhancement [1], thereby allowing additional improvement in shiftability. We prove an important theoretical aspect of the new method: the amplitude spectrum of the length N filter converges almost everywhere to the ideal complex half-band amplitude spectrum as $N \rightarrow +\infty$, thereby assuring shiftability.

1. INTRODUCTION

In this work we describe a new method for generating a filter that we use for generating discrete-time analytic (DTA) signals. Filter characteristics and convergence issues are treated. The filter, as also the DTA signal generated have a number of properties that either equal or exceed those by other methods.

Where the context is clear, we will henceforth use the term "analytic" signal to refer to DTA signals. Methods currently used for generating analytic signals may generally be classified as belonging to one of two classes: either the signal is operated on directly in the frequency domain or filters are involved for their generation. In the former class we have the method of [6], [5] and its extension [1]. Methods based on filter design generally fall under one of two procedures: In the first, a lowpass half-band FIR filter [3], [8] is designed and the spectrum shifted to the right by $\pi/2$, to form a complex half-band filter. Output of these filters provide the analytic signal. The second procedure uses Hilbert transformers [7] to generate FIR filters, which are then used to form the imaginary part of the analytic signal. Adding a delayed version of the real signal provides the analytic signal. Finally, pairs of IIR filters [[7], p.795] in quadrature have also been used to generate the real and imaginary parts of the analytic signal.

In the generation of DTA signals, we seek some desirable properties: The analytic signal should be "invertible" and orthogonal. That is, it should be possible to recover the original real signal from the analytic signal and the real and imaginary parts should be orthogonal. For the filter, desirable properties are generalized linear phase and real-time implementability. For our purposes here, we shall for convenience make reference to all four properties - invertibility, orthogonality, linear phase and real-time - all in the context of the analytic signal or filter itself, rather than prescribing them individually to the DTA signal or the filter. We also consider algorithm efficiency. Lastly and critically, we need to measure how well the spectrum of the analytic signal approximates the ideal one-sided spectrum.

We provide a brief background on the existing techniques in Section 2. The new method is described in Section 3. Filter frequency response is determined in Section 4 and convergence established in Section 5. Properties of the filter are established in Section 6. Experimental results and conclusions are given in Sections 7 and 8 respectively.

2. BACKGROUND

The frequency domain method [6], [5] for generating DTA signals is implemented by the Matlab7 algorithm *hilbert*. Discrete Fourier transform (DFT) coefficients of the given real signal are modified in a simple way: Discrete negative frequency terms, that is, those lying in $-\pi < \omega < 0$ are set to zero. Discrete positive frequency terms lying in $0 \leq \omega \leq \pi$ are multiplied by 2, except those at the end-points $0, \pi$, which are multiplied by 1. The other frequency domain technique [1], referred to as *ehilbert* extends the standard frequency domain method: The continuous spectrum (the Discrete-time Fourier transform (DTFT)) of the real signal is forced to have one additional zero in the negative frequencies. Increasing attenuation in the region $\omega \in (0, -\pi)$ further reduces the single-sideband bandwidth and consequently decreases aliasing under decimation. With respective to the aforementioned properties, the DTA signal generated by *hilbert* is orthogonal and invertible. It is not real-time since the entire signal is needed before generation of the DTA signal. With *ehilbert*, orthogonality of *hilbert* is lost.

In the first of filter design procedures, we consider gen-

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eration of lowpass filters: An optimal lowpass filter is designed and the spectrum shifted by $\pi/2$ to the right to generate a complex half-band filter. In [3] the lowpass filter used is an "optimal" Daubechies scaling filter. This transform is invertible and real time, but does not have linear phase and lacks orthogonality. The second example is that in [8]. Here, the FIR, lowpass filter is an equiripple design. This transformation is invertible (this can be proven using proposition [[3], Sec.2.2, Proposition 1]), real time, linear phase and orthogonal. The second filter design procedure uses Hilbert transformers for the generation of analytic signals. The first method approximates the Hilbert transformer using the Fourier series representation and a Kaiser window. The other uses an equiripple approximation. In both these cases, we have invertibility, real-time implementation, linear phase and orthogonality. Lastly, using IIR filter pairs [[7], p.795], does not give invertibility.

3. THE NEW METHOD

In this paper, we propose a simple procedure for filter design, using the frequency domain method *hilbert*. It possesses all the four properties of invertibility, orthogonality, generalized linear phase and real-time implementation. It is easily scalable and stable. We prove that the frequency response converges almost everywhere to the ideal response as $N \to \infty$. The design is also efficient in that for even N, it has the smallest number of non-zero coefficients. The degree of attenuation in the region $(-\pi, 0)$, or equivalently, the aliasing generated when the filter output is downsampled, is determined by the shiftability [10] property. Lastly, shiftability can always be improved upon using *ehilbert* [1].

We design a filter such that the output to a real signal is an analytic signal. We start with an impulse function of length N, shifted by $N/2$, $((N-1)/2)$ when the length of the desired filter is even, (odd respectively). The choice of the shift assures generalized linear phase. This delayed impulse is operated on by the function *hilbert* [6] to generate the corresponding DTA signal. These signal coefficients constitute the proposed DTA filter. We will prove that as $N \to \infty$, the DTFT amplitude of the DTA filter converges almost everywhere to the function

$$G(e^{j\omega}) = \begin{cases} 2, & 0 < \omega < \pi \\ 0, & -\pi < \omega < 0 \\ 1, & \omega = 0,\ \omega = \pi. \end{cases} \quad (1)$$

To establish behavior for large N we first need to determine the expression for the frequency response of the N-length DTA filter. Convergence is then established for $N \to \infty$.

3. EXPLICIT FORMULA OF THE DTFT OF THE NEW DTA FILTER

3.1. Case when N is a multiple of 4

Let $x(n)$ be the discrete impulse of length N, where N is a multiple of 4, shifted by $N/2$. i.e.,

$$x(n) = \begin{cases} 1, & n = N/2 \\ 0, & 0 \le n \le N-1,\ n \ne N/2. \end{cases}$$

Applying algorithm [6] to $x(n)$, we obtain the expression of the DTA signal $z(n)$, [[1], Eqs. (1), (2)]. For n even, we have

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p+1)\cot(\pi(n-(2p+1))/N),$$

and for n odd

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p)\cot(\pi(n-2p)/N).$$

$z(n)$ constitutes the impulse response of the DTA filter. We observe the efficiency of the representation here. The only non-zero even element of $x(n)$ is at $n = N/2$, which is an even index, and equal to 1. Therefore,

$$z(n) = \begin{cases} 0, & n \ \text{ even},\ n \ne N/2 \\ 1, & n = N/2 \\ j(2/N)\cot(\frac{\pi}{N}n - \frac{\pi}{2}), & n \ \text{ odd}. \end{cases} \quad (2)$$

3.2. Case when N even and $N/2$ odd

Similarly to the above, for the case N even and $N/2$ odd, the impulse response of the DTA filter is given by

$$z(n) = \begin{cases} j(2/N)\cot(\frac{\pi}{N}n - \frac{\pi}{2}), & n\,\text{even} \\ 0, & n\,\text{odd},\ n \ne N/2 \\ 1, & n = N/2. \end{cases} \quad (3)$$

In this case also we observe that the only non-zero odd element of $x(n)$ is at $n = N/2$, which is an odd index, and equal to 1. Therefore we conclude that for the case N even, about half of the DTA filter coefficients are equal to zero.

Formulae for N odd are given in [[2], pp. 39-43]. The derivation of the DTFT $Z(e^{jw})$ of $z(n)$ is long and we provide only the main results here. Details can be found in [[2], pp. 34-44].

$$Z(e^{j\omega}) = \begin{cases} e^{-j\omega N/2}\{1 + \frac{4}{N}\sum_{n=0}^{\frac{N}{4}-1}\tan(\frac{\pi}{N}(2n+1)) \\ \sin(\omega(\frac{N}{2} - (2n+1)))\},\ \text{for } N\,\text{a multiple of 4} \\ \\ e^{-j\omega N/2}\{1 + \frac{4}{N}\sum_{n=1}^{\frac{1}{2}(\frac{N}{2}-1)}\tan(\frac{\pi}{N}2n) \\ \sin(\omega(\frac{N}{2} - 2n))\},\ \text{for } N\,\text{even } N/2\,\text{odd} \\ \\ e^{-j\omega\frac{N-1}{2}}\{1 + 2/N\sum_{n=1}^{\frac{N-1}{2}}\frac{\cos(\frac{\pi n}{N}) - (-1)^n}{\sin(\frac{\pi n}{N})} \\ \sin(n\omega)\},\ \text{for } N \ \text{odd}. \end{cases} \quad (4)$$

4. CONVERGENCE

The closed form expression $Z(e^{j\omega})$ of the N-point DTA filter is used to establish convergence for $N \to \infty$. We can show that its amplitude converges almost everywhere to the function in equation (1).

The proof is available for $N = 2^p$, where p is a positive integer. Other cases, (i) N is even and $N/2$ odd and (ii) N is odd, are presently under investigation.

Theorem 1: The function

$$|H(e^{j\omega})| = |1+\frac{4}{N}\sum_{n=0}^{\frac{N}{4}-1}\tan(\frac{\pi}{N}(2n+1))\sin(\omega(\frac{N}{2}-(2n+1)))|$$

converges almost everywhere to $G(e^{j\omega})$ given by equation (1), when $N \to +\infty$ and $N = 2^p$, where p is a positive integer .

Proof: The proof can be found in [[2], pp. 45-47].

5. PROPERTIES OF THE NEW DTA FILTER

5.1. Generalized Linear Phase

Theorem 2: The DTA filter has generalized linear phase.

Proof: The proof follows from equation (4).

5.2. Orthogonality

Theorem 3: The real and imaginary parts of the DTA filter are orthogonal.

Proof: The proof can be found in [[2], pp. 47-49].

5.3. Shiftability

Theorem 4: The DTA filter realizes shiftability.

Proof: By Theorem (1), we conclude that in the limit as $N \to \infty$, the frequency response of the DTA filter approaches the ideal response. Hence the DTA filter realizes shiftability.

6. EXPERIMENTAL RESULTS

A basic issue in the design of DTA filters is in the "goodness" of approximation to the ideal. While this may be measured in many ways, such as mean-square error, we consider measuring it here in the context of a particular application. That is, we measure the aliasing generated when the corresponding DTA signal is subsampled. Such applications occur in the design of complex wavelets [9]. Aliasing is conveniently measured using the property of shiftability [10].

Shiftability is seen as follows: In the critically sampled wavelet transform, translation of the input signal leads to transform coefficient energy moving both within and across subbands. However, it is possible that the information contained within a subband remain within the subband after signal translation. Necessary and sufficient conditions for this is the Nyquist criterion [10]. Accordingly, subsampling leading to a variation in subband-energy reflects the presence of aliasing.

We compare shiftability of the DTA filters with that by the new method. This involves applying an impulse to a DTA filter to generate an analytic impulse. This is applied to a critically sampled wavelet filter bank. Coefficient energy at each subband level is measured as the input impulse is cyclically translated. Constant subband-energy implies shiftability.

In our experiments we use a Daubechies'D_8 filter bank at M-levels (*wfilters('db8')* in Matlab7). This particular filter bank is chosen since it has been shown [[3], Sec.3.3, Proposition 1] that using the Strang-Nguyen model [[11], p. 172] for the Daubechies'D_8 scaling filter, and using a Daubechies DTA filter of length N satisfying the conditions of Proposition 1, shiftability is obtained. The value $N = 28$ was used. Accordingly we compare performance of all our DTA filters using this model: A Daubechies'D_8 critically sampled wavelet filter bank, DTA filters of size $N = 28$ and $M = 3$. Where filter sizes need to be odd, we use $N = 27$.

DTA filters were generated by all five methods: The first one was a lowpass Daubechies (LP_daub) scaling filter (Daubechies'D_{14} filter) of length 28 shifted by $\pi/2$ [3]. Another lowpass filter [8] (LP_equiripple) of length 27 and transition band width of 0.16π was generated using the function *firpm* in Matlab7. Two complex half-band filters of length $N = 27$ were generated using Hilbert transformers. The first (HT_equiripple) [[7], pp. 792-794] was designed using the function *firpm* with the option *Hilbert* and the passband frequency range $[0.1\pi, 0.9\pi]$. The second (HT_windowed) [[7], p.795] was the Fourier series approximation with a *kaiser* window using $\beta = 3.227$. This was derived taking the minimum attenuation between 21 and 50 dB. For the new method, a length 27 complex half-band filter was generated. (An impulse of length 28 was used. Since the first element of the DTA is zero, by equation (2), we consider the length as 27).

The five DTA filters were used to generate analytic impulses. Subband-energy was measured at the 3 levels of a critically sampled discrete wavelet transform (Daubechies' D_8 filter) over 16 circular shifts of the input impulse. Figure (1) shows the transform subband-energy at different scales, as a function of input signal shifts. Results for [3] are not shown since they did not compare favorably. At level-1 highpass, we observe that the new method out performs all others. Better shiftability is also attained at the two other subbands, although by not as large an amount.

For a quantitative measure of performance, we measure the variation of subband-energy using the concept of total variation. The latter is defined as $TV = \sum_{n=1}^{16} |x_k - x_{k-1}|$, where x_k represents subband-energy. We use the ratio of TV_new method and TV_other method for comparison. Results are shown in Table 1, where the small numbers indicate much superior performance with the new method.

Improved shiftability, that is a further reductions in aliasing, can be obtained by applying *ehilbert* [1] instead of *hilbert* to the delayed input impulse. We choose to zero the negative spectrum at an additional negative frequency $\omega = -2.0415$. We use the ratio TV_ehilbert and TV_hilbert for comparison. Results are shown in Table 2. We see that with the new method we have reduced aliasing in all levels except that at Level-2 highpass where the results are close.

The new method is efficient since for the case N even, about half the filter coefficients are equal to zero, as shown

Methods	Level-1 highpass	Level-2 highpass
LP_daub	0.00227	0.80658
HT_windowed	0.13275	0.87737
HT_equiripple	0.09366	0.87185
LP_equiripple	0.01359	0.86547

Methods	Level-3 highpass	Level-3 lowpass
LP_daub	0.67797	0.22899
HT_windowed	0.86296	0.71851
HT_equiripple	0.84548	0.65256
LP_equiripple	0.83910	0.64381

Table 1: Comparison of the total variation (TV) at each level.

$\frac{\text{TV of DTAF using } ehilbert}{\text{TV of DTA using } hilbert}$	Subband level
0.78302	Level-1 highpass
1.01261	Level-2 highpass
0.98746	Level-3 highpass
0.77648	Level-3 lowpass

Table 2: Comparison of the total variation (TV) at each level.

in equations (2), and (3). The only other method where this efficiency also occurs is in HT_windowed. We note that the HT_equiripple design using the Matlab7 function *firpm* is still not stable for large N, as also concluded in [8]. This is due to the recursive algorithm which causes rounding machine errors. The new method is stable for all N.

7. CONCLUSION

We have proposed a new method for generating a DTA signal using the function *hilbert*. This method satisfies invertibility, orthogonality, linear phase and real-time implementability. Its shiftability property and efficiency, are equal to or better than those from all other methods. It is stable for large N. Further improvement in shiftability can be obtained using the extension *ehilbert*.

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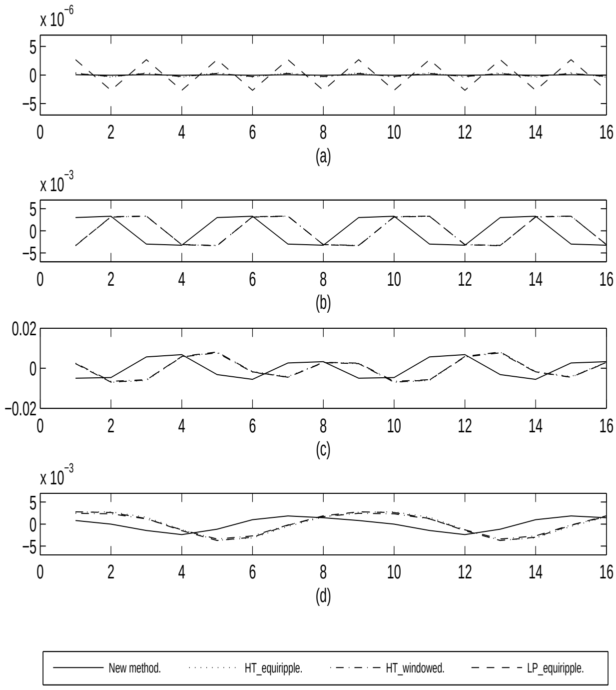

Figure 1: Comparison of shiftability for the three methods. (a) Level-1 highpass subband-energy,(b) Level-2 highpass subband-energy,(c) Level-3 highpass subband-energy,(d) Level-3 lowpass subband-energy.

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