WREATH PRODUCT CYCLIC GROUP-BASED CONVOLUTION: A NEW CLASS OF NONCOMMUTATIVE FILTERS

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ABSTRACT

The theory of spectral analysis of a particular class of noncommutative groups—wreath products of cyclic groups—has been shown to have a group-based convolution that leads to a new class of noncommutative filters. These filters, with their group and scale-selective properties and their relationship to DFT filter banks, offer some intriguing possibilities in signal processing applications. In this paper we give a brief summary of some of the basic properties of convolution with wreath product cyclic groups and illustrate those properties through an example. Applications to some basic signal processing tasks are proposed.

1. INTRODUCTION

The theoretical basis for the group-based approach to signal processing and the resulting convolution and correlation properties, have been established and reported in papers: Part I [2] and Part II [3]. That body of work had its origin in earlier work [1] where the possibility of using wreath product cyclic (WPC) groups for signal processing was introduced and the use of WPC group-based filters as a correlators for detection of similar patterns was broached. This paper extracts from [2] and [3], where full details may be found, some of the essential features of WPC group-based convolution, emphasizing their signal processing and filtering characteristics.

We give a brief summary of our earlier results: In Part I [2], for wreath product groups that arise as symmetries of spherically homogeneous trees (WPC groups), we determined the spectral representation of signals defined on the leaves of a tree. These representations gave a (non-abelian) generalization of the discrete Fourier transform (DFT), for which the 1-D and 2-D DFT are special cases (when the tree has only one level). We saw also that the multichannel DFT filter bank as well as the Haar wavelet transform followed naturally from our WPC group representation; hence we obtained a multiresolution DFT. When all the cyclic factors have the same order 4^k, the so-called quadtree WPC group spectrum of an image was determined and displayed and related to a one-dimensional unitary block transform. In Part II [3], we saw a generalization of discrete cyclic convolution to group-based convolution for arbitrary groups. Finite group-based convolution was defined and spatial and spectral properties established. We paid particular attention to WPC group-based convolution, generating a general formula for WPC convolution in terms of δ-functions. Application to a problem in similarity determination was examined. Group-based correlation was defined and seen to extend the peak-matching property of standard correlation to images that were group transformations of each other.

In Section 2 we define group-based convolution and show its determination in the spectral domain. A brief description of scale selective properties of convolution are given in Section 3. In Section 4 we give an application to a simple problem in similarity determination. We conclude with some ideas about where these filters may find other applications.

2. CONVOLUTION OVER WPC GROUPS

Let G be a finite group acting on a set $X = \{x_0, x_1, \ldots, x_{N-1}\}$. Let $L(X) = \{f \mid f : X \to \mathbb{C}\}$. For discrete signals f and h of length N (i.e., $f, g \in L(X)$), their cyclic convolution is

$$(f \star h)(x_n) = \sum_{m=0}^{N-1} f(x_m)h(x_{n-m}) = \sum_{m=0}^{N-1} f(\sigma^m x_0)h(\sigma^{-m} x_n), \tag{2.1}$$

where σ is the N–cycle that cyclically permutes the set $x_0, x_1, \ldots, x_{N-1}$, and the underlying group G is the cyclic group of order N generated by σ. The formula for group-based convolution over an arbitrary group G acting on X arises as a natural generalization of the cyclic convolution. That is, group-based convolution of signals f and $h \in L(X)$, written henceforth as $f \star h$, is defined as

$$(f \star h)(x_n) = \frac{|X|}{|G|} \sum_{\beta \in G} f(\beta x_0)h(\beta^{-1} x_n). \tag{2.2}$$

As with continuous and discrete cyclic convolutions, group-based convolution enjoys many properties (linearity, associative, etc.,) which are easily understood by passing to the spectral domain.

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Let $X_{\mathbf{m}}$ be the set of leaves of the spherically homogeneous rooted tree $\mathcal{T}_{\mathbf{m}}$ and let G now be the WPC group $Z_{\mathbf{m}}$ acting on the leaves of the tree $X_{\mathbf{m}}$. In this case $f \star h$ defined by (2.2) is called *WPC convolution*. Since WPC convolution may require as many as $|\,G\,|$ multiplications to compute each value, this method can be computationally prohibitive even for small trees. It is computationally more effective to pass to the spectral domain, achieved as follows. Let $Q(k,n)$ be the quadtree with n nonzero levels, $Z(k,n)$ the WPC group acting on $Q(k,n)$ and $L(k,n)$ the set of functions defined on the leaves of $Q(k,n)$. When $G = Z(k,n)$ and signals $f,\ h \in L(k,n)$ have a quadtree spectrum $\mathcal{Q}(f)$ and $\mathcal{Q}(h)$ respectively (cf. Section 5.1 in [2]), we have the following result for computing convolution:

Theorem 2.1 Convolution of signals f and h in $L(k,n)$ may be computed in the spectral domain by multiplying each entry in each nested grid irreducible submatrix of spectrum $\mathcal{Q}(f)$ by the upper lefthand entry of the corresponding block matrix of spectrum $\mathcal{Q}(h)$.

Example. Following the notation of Example 1 in Section 5.1 in [2], if f and h are in $L(1,2)$, denote $\mathcal{Q}(f)$ and $\mathcal{Q}(h)$ by the 4×4 matrices $(a_{i,j})$ and $(b_{i,j})$ respectively. Then

$$\mathcal{Q}(f) \cdot \mathcal{Q}(h) = \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,3} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,3} \\ a_{3,1}b_{3,1} & a_{3,2}b_{3,1} & a_{3,3}b_{3,3} & a_{3,4}b_{3,3} \\ a_{4,1}b_{3,1} & a_{4,2}b_{3,1} & a_{4,3}b_{3,3} & a_{4,4}b_{3,3} \end{pmatrix}$$

This shows that WPC convolution is not generally commutative, nor unique in the right variable h.

3. WPC CONVOLUTION AND SCALE-SELECTIVE FILTERING

We analyze the effect of WPC convolution of images interpreted as functions on the leaves of a regularly branching tree (cf. Section 3.2.2 of [2]). We impart to a 512×512 image the structure of a quadtree $Q(1,9)$ acted on by $Z(1,9)$ as follows: Let Q^0 denote the whole image and label the four 256×256 subquadrants in a clockwise fashion as $Q_0^1, Q_1^1, Q_2^1, Q_3^1$ respectively. Proceeding recursively, Q_i^k is the $2^{9-k} \times 2^{9-k}$ subimage where $i = 0,1,2,3$ and $k = 0,1,2,\ldots,8$. $f \star h$ then consists of a superposition of cyclic convolutions and averaging operations. All 4-element sequences of f are cyclically convolved in-place with the 4-element sequence of h defined on quadrant Q_0^8. Let f_C be the signal supported on the first four leaves of $Q(1,9)$. A filter h supported on quadrants Q_j^k and convolved with f_C generates weighted averages of f_C on the same quadrants Q_j^k, for $k = 8,7,\ldots,1$ and $j = 1,2,3$. The remaining elements of f–which are of the form $\tau(f_C)$ for some $\tau \in G$–when convolved with h appear as rotated (by τ) versions of the operations on f_C. Hence, we observe that the convolved output $g = f \star h$ appears as a pattern in blocks of size 4^k, for $k = 0,1,\ldots,8$; cyclic convolution effects appear in all 4-element sequences ($k = 0$), while the weighted averages appear in sequences of increasing size 4^k, for $k = 1,2,\ldots,8$. Let h_0 be the filter having support on the $(1,1)$ position in Q_0^8 and h_i as that with support on the $(1, 1+2^{i-1})$ position in Q_1^{10-i}, $1 \le i \le 9$. We define *extensions* of h_i $1 < i \le 9$, as filters with a unit impulse located anywhere in quadrant Q_1^{10-i}. Also, unit impulses supported in the upper lefthand 2×2 block are called *extensions* of h_0 and f_C. We then have:

Theorem 2.2

(1) Filter h_0 and its extensions cyclically convolve in-place blocks of f of size 2×2.

(2) For $i = 2,3,\ldots,9$, filters h_i and their extensions act as scale-selective filters, and they average f_C over a block of size $2^{i-1} \times 2^{i-1}$. The averaging results g_i are supported in subquadrants Q_1^{10-i}, the same as the filter, thereby reflecting a local rotation of the image. Group operations on h_i acting at level $9-i$ result in the corresponding group operation on the outputs $g_i = f_C \star h_i$, for $i = 2,3,\ldots,9$.

(3) WPC convolution $f \star h$ of arbitrary images is then obtained using transformation of convolution under group operations on f_C and h_i and bilinearity.

4. WPC CONVOLUTION AND SCALE SIMILARITY

While the correlation coefficient serves as a good indicator for measuring linearly correlated signals, it does not perform well for images at different scales or with rotation, distortion or texture differences. In such cases perceptually similar images with low correlation values may imply a mismatch and thus vitiate the advantages of normalized correlation. To address the problem of low correlation for images at different scales, we investigate the effects of measuring the correlation of images transformed to a WPC representation, where perceptually similar images can become measurably closer. The technique of applying local linear transformations has been adopted by others, often utilizing some application of the Karhunen-Loéve transform. Here we attempt to match images, especially those generated at various scales, by first transforming them to a multiresolution representation and then using normalized correlation. For determining similarity of a scaled image f_i to a prototype image f we compare $f_i \star f$ and $f \star f$ using standard correlation and compare that to the standard correlation of f_i and f. We carry out this analysis by utilizing the multiresolution representation: for the tree $Q(1,9)$, using the decomposition $f = \sum_{j=1}^{9} f^j$, where f^j is f defined over quadrants Q_0^8 and $Q_1^i + Q_2^i + Q_3^i$, for $i = 8,7,\ldots,1$, we have seen that WPC convolution of signal g with f is the superposition of signals constant over successive lengths of size 4^k, for $k = 0,1,\ldots,8$. Specifically, $f_i \star f$ gives a weighted decomposition of f_i consisting of a cyclic convolution and the sum of weighted by f^j averages of f_i.

Example 1. We see in Figures 1(a) and (b) two 256×256 test images f_1 and f_2 obtained from the WPC convolution of a prototype image f with specific filters h_{LP_1} and h_{HP_2}. Accordingly, f_1 and f_2 are respectively, projections of f at scale 1 lowpass and of f at scale 2 highpass. We attempt to measure similarity of both to f.

Consider the lowpass image f_1 first. By construction f_1 is dissimilar to f at scale 1 highpass where its spectrum is zero, but similar at all other scales. Accordingly, the spectrum of $f_1 \star f$ is similar to that of $f \star f$ at all scales except scale 1 highpass, where the former has zero spectrum. Consequently, all successive four-point sequences in $f_1 \star f$ will be constant. Since their sum is kept the same as in the corresponding part of signal $f \star f$, the spectrum of the two convolutions is the same at all other scales. The resulting effect is that the two convolution patterns will be similar at all scales except scale 1 highpass, where $f_1 \star f$ will be constant. We observe the general similarity in Figure 1(c). The dissimilarity at scale 1 is seen by observing the 4-point sequences in Figure 1(e). Here we observe that $f_1 \star f$ is constant in all successive 4-point sequences while $f \star f$ is not. Similarity at scales greater than 1 is manifested by the fact that $f_1 \star f$ has the same frequency characteristics as $f \star f$ for sequences considered in successive blocks of length 4^k, for $k = 1, 2, \dots, 7$.

We devise a bandpass image f_2 similar to f at scale 2 highpass but dissimilar at others, where it has zero spectrum. Spectra of the resulting convolutions $f_2 \star f$ and $f \star f$ carry those same characteristics. Figure 1(d) shows the two convolutions, which clearly appear dissimilar. Examining convolutions at the scale of interest, namely scale 2, we see two typical 16-point sequences in Figure 1(f). We observe that both convolutions have the same frequency characteristics at that scale, that is, for successive segments of length 4^2, taken in blocks of size 4. Dissimilarity at scale 1 is discernible. Dissimilarity at other scales exists due to $f_2 \star f$ having a zero sum over successive sequences of length 4^k, for $k = 2, 3, \dots, 7$.

We now apply standard normalized correlation to measure signal similarity. Before WPC transformation the correlation coefficient, ρ, of f_i and f is 0.9421 and 0.3817 for $i = 1, 2$ respectively. After WPC convolution, as described above, it is 1 and 0.0019. A corresponding effect holds for f_i similar to f at the other lowpass and highpass scales. Hence for this example and for the lowpass case, transformation using WPC convolution provides greater normalized correlation than that obtained without convolution.

A correlation coefficient of 0.3817 is not high, and would typically not imply a good match. For the WPC case, since the low value of 0.0019 is due to matching at only 1 scale and a mismatch at all others, it would appear logical to compare convolutions at each scale rather than simultaneously across all scales. Accordingly, we compare $f_2 \star h_{HP_i}$ and $f \star h_{HP_i}$ for $i = 1, 2, \dots, 8$, where h_{HP_i} are the highpass filters at the various scales. Using covariances, since the sum of the values of the projections of f_2 are zero for $i = 1, 2, \dots, 8$, we get, as expected, covariance of value 0 at all scales except scale 2 where ρ is 1. Hence we establish high correlation for this highpass signal using projections at each scale. Accordingly, convolving to obtain a multiresolution representation for the lowpass case, and at each scale as in the highpass example, results in a characterization that generates higher correlation values than those obtained directly. A more general example can be seen in [3].

5. CONCLUSION AND OPEN RESEARCH

WPC convolution of images f and h defined by the quadtree structure as described above, consists of cyclic convolutions and weighted averages of 2×2 blocks of f, by elements of h. This can lead to scale-selective filtering. In the example given, we see that WPC convolution establishes an averaging type of multiresolution decomposition that can be utilized for solving some problems in the determination of similarity of signals.

Some key features of our group-based theory that are of interest in signal processing are the multiresolution spectrum and the convolution operation. For WPC groups where the associated tree is a quadtree, their spectral representation yields familiar structures like the multichannel pyramid DFT filter bank and the 2-D Haar transform. Also, new operations like convolution which can be obtained by spectral multiplication and which has a scale-selective property. Additionally, both the spectrum and convolution are distinguished by a group-invariance property. Hence, group-based theory, and specifically WPC group-based spectral analysis, appear to have some significant potential for application in signal processing. A few key problems worth investigating would be the following:

WPC convolution and similarity determination: We have seen that transforming signals using WPC convolution can be useful in establishing similarity of certain types of images, while still maintaining a discrimination capability. A quantitative treatment of similarity determination, classification accuracy, and scalability is needed.

WPC convolution and matched filtering: What is the signal-to-noise ratio for detection of known signal-in-noise for the WPC class of linear *noncommutative* filters? How does that compare with the commutative case? What process does the WPC transform diagonalize?

Extensions to 2-D: Given the one-dimensional unitary WPC transform, a two-dimensional separable unitary transform can be defined; however, its group theoretic implications are not presently clear. Does there exist such a theory? If so, what are its implications, especially those analogous to the one-dimensional case?

References

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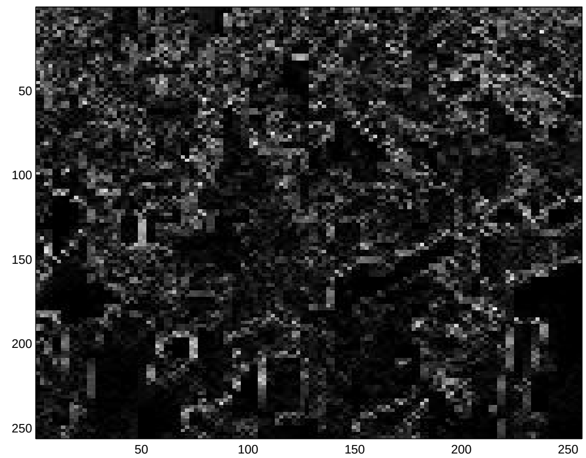
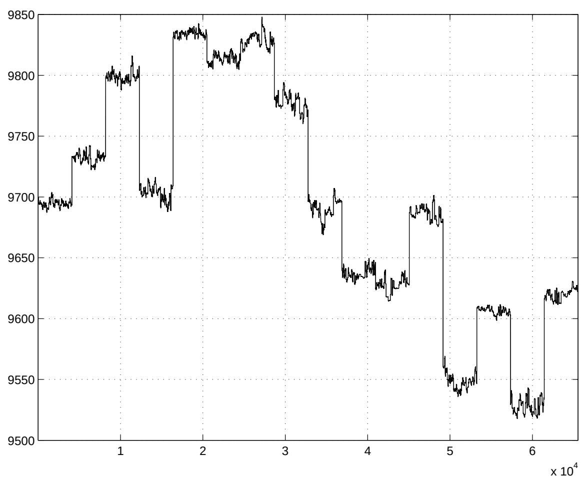
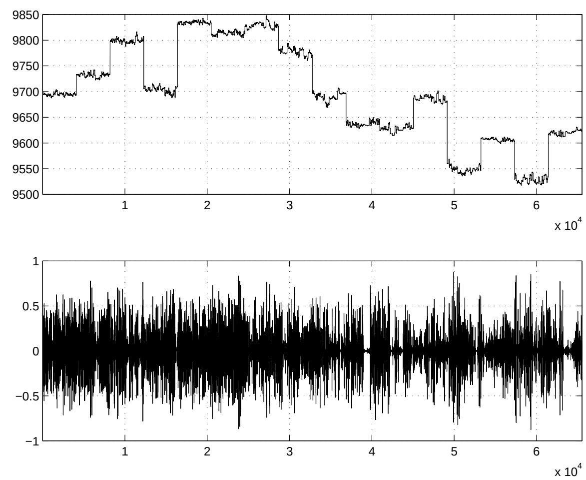
(a) Lowpass image f_1 at scale 1



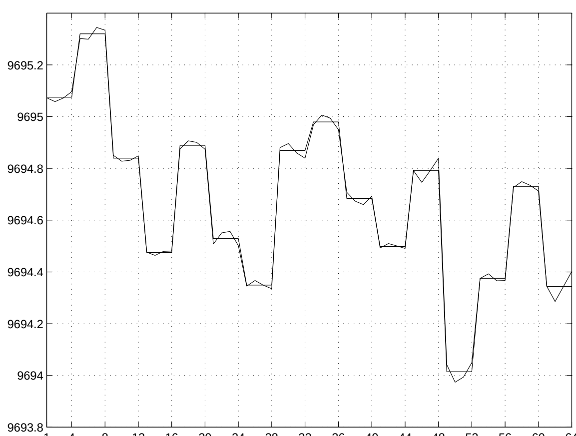
(b) Highpass image f_2 at scale 2



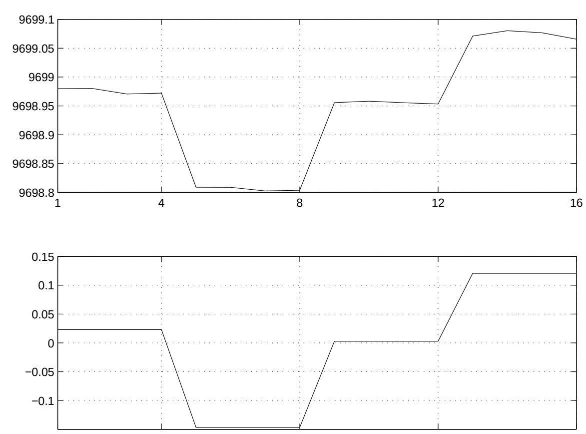
(c) $f \star f$ and $f_1 \star f$



(d) $f \star f$ and $f_2 \star f$



(e) $f \star f$ and $f_1 \star f$ - zoomed



(f) $f \star f$ and $f_2 \star f$ - zoomed

Figure 1: WPC convolution using WP filtered images