Section 12.2

Definition: A differential equation is separable if we can write it as y' = f(x)g(y).

We can separate it into $\frac{1}{g(y)}y' = f(x)$

We can solve this by integrating both sides with respect to their appropriate variables. So we have

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

Example: Solve y' = y/x

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln y = \ln x + c$$

$$\ln y = \ln x + \ln C = \ln(xC)$$

$$\ln y = \ln(Cx)$$

$$y = Cx$$

Example: Solve

$$\frac{dy}{dx} = \frac{y^2}{x+1}$$

$$y^{-2}\frac{dy}{dx} = \frac{1}{x+1}$$

$$\int y^{-2}\,dy = \int \frac{1}{x+1}\,dx$$

$$-y^{-1} = \ln(x+1) + c$$

$$-\frac{1}{y} = \ln(x+1) + \ln C$$

$$\frac{1}{y} = -\ln[C(x+1)]$$

$$y = \frac{-1}{\ln[C(x+1)]}$$

Example: Solve $y' = \frac{xy^2 + x}{y}$, $y(0) = 0$

Solve

$$y' = \frac{xy^2 + x}{y} = \frac{x\left(y^2+1\right)}{y}$$

$$\frac{y}{y^2+1} y' = x$$

$$\int \frac{y}{y^2+1}\, dy = \int x\, dx$$

$$u = y^2 + 1$$

$$du = 2y\, dy$$

$$\frac{1}{2}\, du = y\, dy$$

$$\frac{1}{2}\int \frac{1}{u}\, du = \int x\, dx$$

$$\frac{1}{2}\ln u = \frac{1}{2}x^2 + c$$

$$\ln\left(y^2+1\right) = x^2 + 2c$$

$$e^{\ln\left(y^2+1\right)} = e^{x^2+2c}$$

$$y^2 + 1 = e^{x^2} e^{2c}$$

$$y^2 = Ae^{x^2} - 1$$

Now, let's use the boundary conditions y(0) = 0

$$y^2 = Ae^{x^2} - 1$$

$$0 = Ae^0 - 1$$

$$0 = A - 1$$

$$A = 1$$

Solution:

$$y^2 = e^{x^2} - 1$$