

Lecture: Sec. 7.4 Improper Integrals

Problem: Find the area between the curve $f(x) = 1/x^2$ and the x-axis for $x > 1$.
From experience, we would anticipate the integral

$$\int_1^{\infty} \frac{1}{x^2} dx.$$

Reminder: The Fundamental Theorem of Calculus states

Let $f(x)$ be continuous on $[a, b]$ and let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

That is, the Fundamental Theorem of Calculus says that we can only use the antiderivative if the function is defined on a closed interval. The example is not. So we call this an **improper integral** and need a new definition.

Definition:

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

where $b > a$, provided that this limit exists. If the limit does not exist, we say that the integral diverges; otherwise, the integral converges.

Examples:

$$\begin{aligned} 1. \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \left[\int_1^b \frac{1}{x^2} dx \right] = \lim_{b \rightarrow \infty} \left[\int_1^b x^{-2} dx \right] = \lim_{b \rightarrow \infty} \left[-x^{-1} \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{-1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{b} - \frac{-1}{1} \right] = \lim_{b \rightarrow \infty} \left[-\frac{1}{b} + 1 \right] = 0 + 1 = 1 \end{aligned}$$

We say that this integral converges to 0

$$\begin{aligned} 2. \int_1^{\infty} \frac{2}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \left[\int_1^b 2x^{-1/3} dx \right] = \lim_{b \rightarrow \infty} \left[2 \left(\frac{3}{2} \right) x^{2/3} \right]_1^b = \lim_{b \rightarrow \infty} \left[3x^{2/3} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[3b^{2/3} - 3(1)^{2/3} \right] = \infty \end{aligned}$$

So this integral diverges.

Definition:

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx, a < b$$

that this limit exists. If the limit does not exist, we say that the integral diverges; otherwise, the integral converges.

Example:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^3}{(x^4+1)^2} dx &= \int_{-\infty}^0 \frac{x^3}{(x^4+1)^2} dx + \int_0^{\infty} \frac{x^3}{(x^4+1)^2} dx \\
&= \lim_{a \rightarrow -\infty} \int_a^0 \frac{x^3}{(x^4+1)^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{x^3}{(x^4+1)^2} dx \\
&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{4(x^4+1)} \right]_a^0 + \lim_{b \rightarrow \infty} \left[-\frac{1}{4(x^4+1)} \right]_0^b \\
&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{4(0^4+1)} + \frac{1}{4(a^4+1)} \right] + \lim_{b \rightarrow \infty} \left[-\frac{1}{4(b^4+1)} + \frac{1}{4(0^4+1)} \right] \\
&= \lim_{a \rightarrow -\infty} \left[-\frac{1}{4} + \frac{1}{4(a^4+1)} \right] + \lim_{b \rightarrow \infty} \left[-\frac{1}{4(b^4+1)} + \frac{1}{4} \right] \\
&= -\frac{1}{4} + 0 - 0 + \frac{1}{4} = 0
\end{aligned}$$

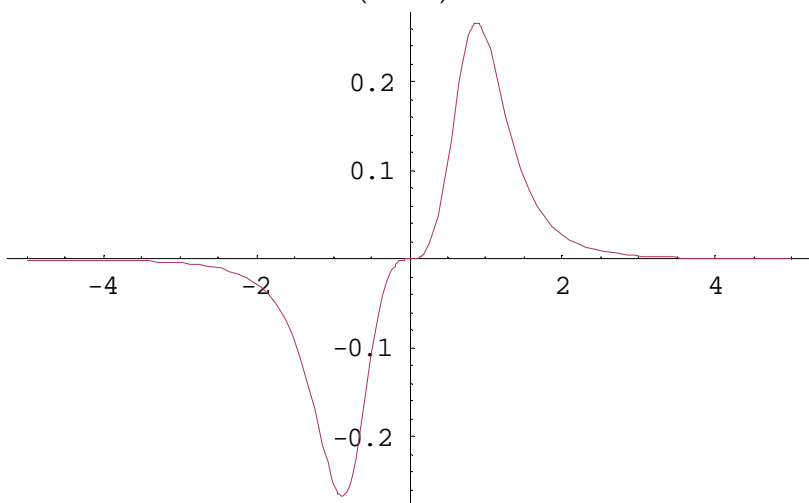
$$\int \frac{x^3}{(x^4+1)^2} dx = \frac{1}{4} \int \frac{1}{u^2} du = \frac{1}{4} \int u^{-2} du = \frac{1}{4} (-1)u^{-1} + C = -\frac{1}{4(x^4+1)} + C$$

$$u = x^4 + 1$$

$$du = 4x^3 dx$$

$$\frac{1}{4} du = x^3 dx$$

Graph the function $f(x) = \frac{x^3}{(x^4+1)^2}$



The two regions have the same area (they are symmetric) but one is positive and the other is negative. Thus, they cancel and the integral would be 0.