Lecture: Sec. 7.4 Improper Integrals

Problem: Find the area between the curve f(x) = 1 / x^2 and the x-axis for x > 1.
From experience, we would anticipate the integral

$$\int_1^{\infty} \frac{1}{x^2}\, dx.$$

Reminder: The Fundamental Theorem of Calculus states

Let f(x) be continuous on [a, b] and let F(x) be any antiderivative of f(x). Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

That is, the Fundamental Theorem of Calculus says that we can only use the antiderivative if the function is defined on a closed interval. The example is not. So we call this an **improper integral** and need a new definition.

Definition: We define the improper integral

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)\, dx$$

where b > a, provided that this limit exists. If the limit does not exist, we say that the integral diverges; otherwise, the integral converges.

Examples:

1. $$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\, dx = \lim_{b\to\infty}\int_1^b x^{-2}\, dx = \lim_{b\to\infty}\left[-x^{-1}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{1}\right] = 0+1 = 1$$

The answer to the initial problem is that the area is 1.

2. $$\int_1^{\infty} \frac{2}{\sqrt[3]{x}} dx = \lim_{b\to\infty}\int_1^b 2x^{-1/3}\, dx = \lim_{b\to\infty}\left[2\left(\frac{3}{2}\right)x^{2/3}\right]_1^b = \lim_{b\to\infty}\left[3b^{2/3} - 3(1)^{2/3}\right] = +\infty$$

So this integral diverges.

Definition:

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, a < b$$

provided that this limit exists. If the limit does not exist, we say that the integral diverges; otherwise, the integral converges.

Example:

$$\int_{-\infty}^{-1} e^{2x}\, dx = \lim_{a\to-\infty}\int_a^{-1} e^{2x}\, dx = \lim_{a\to-\infty}\left[\frac{1}{2}e^{2x}\right]_a^{-1} = \lim_{a\to-\infty}\left[\frac{1}{2}e^{-2} - \frac{1}{2}e^{2a}\right] = \frac{1}{2}e^{-2}$$

$$\lim_{a\to-\infty} e^{2a} = 0$$

$$\int_{-\infty}^{\infty} \frac{x^3}{\left(x^4+1\right)^2} dx = \int_{-\infty}^{0} \frac{x^3}{\left(x^4+1\right)^2} dx + \int_{0}^{\infty} \frac{x^3}{\left(x^4+1\right)^2} dx = -\frac{1}{4} + \frac{1}{4} = 0$$

$$\int_{-\infty}^{0} \frac{x^3}{\left(x^4+1\right)^2} dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{x^3}{\left(x^4+1\right)^2} dx = \lim_{a \to -\infty} \left[\frac{-1}{4\left(x^4+1\right)} \right]_a^0$$

$$= \lim_{a \to -\infty} \left[\frac{-1}{4\left(0^4+1\right)} - \frac{-1}{4\left(a^4+1\right)} \right] = -\frac{1}{4} + 0 = -\frac{1}{4}$$

$$\int_{0}^{\infty} \frac{x^3}{\left(x^4+1\right)^2} dx = \lim_{b \to \infty} \int_{0}^{b} \frac{x^3}{\left(x^4+1\right)^2} dx = \lim_{b \to \infty} \left[\frac{-1}{4\left(x^4+1\right)} \right]_0^b$$

$$= \lim_{b \to \infty} \left[\frac{-1}{4\left(b^4+1\right)} - \frac{-1}{4\left(0^4+1\right)} \right] = 0 + \frac{1}{4} = \frac{1}{4}$$

$$\int \frac{x^3}{\left(x^4+1\right)^2} dx = \frac{1}{4} \int u^{-2}\, du = \frac{1}{4}(-1)u^{-1} + C = -\frac{1}{4\left(x^4+1\right)} + C$$

$$u = x^4 + 1$$

$$du = 4x^3\, dx$$

$$\frac{1}{4} du = x^3\, dx$$