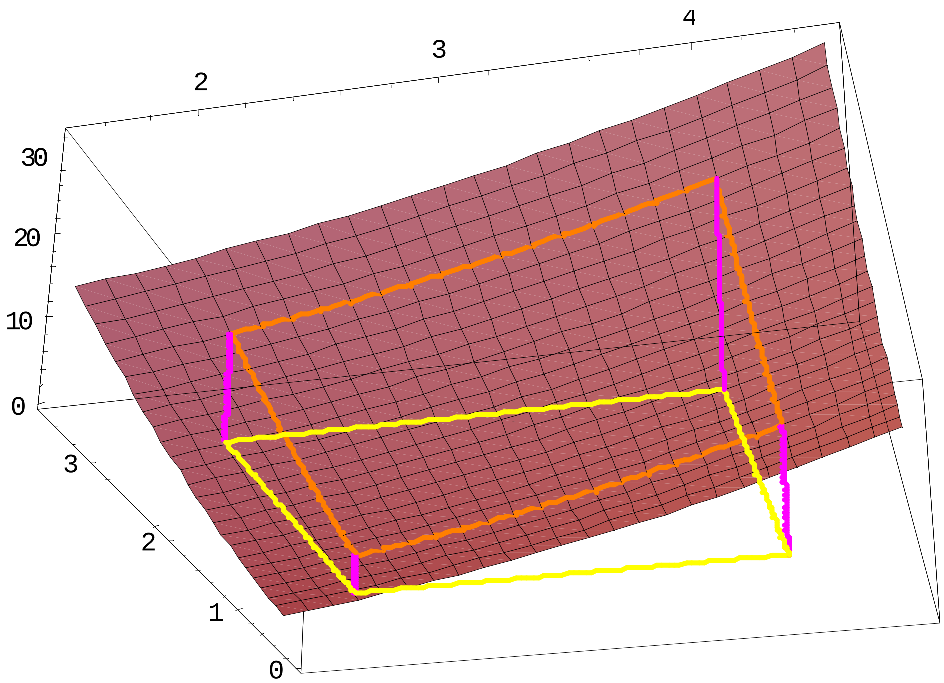
March 15, 2006
Math 20D

Section 8.6

Problem: Find the volume of the solid which has the function f(x, y) = $x^2 + y^2$ as the top and is above the rectangle bounded
by the lines x = 1, x = 3, y = 2 and y = 4.



Yellow lines are the base of the solid and have z = 0. The x lines are horizontal and the y lines are diagonal. The magenta lines go from the corners of the base up to the surface. The orange lines are on the surface and "match" the yellow lines.

We are looking for the volume of the "box" which does not have a flat top. If the top were flat, we could use the formula length x width x height.

Now we are going to divide the base of this "box" into little rectangles and build boxes on top; we will use the value of the function at some point in each little box to approximate the height of the box. So, if the width of each box is Δx and the length is Δy and height is f(x, y), then the volume would be f(x,y) $\Delta x \Delta y$.

$$V = \iint_R f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

$f(x,y) = x^2 + y^2$

R is the base which, in this case, is a rectangle whose sides are x = 1, x = 3, y = 2, and y = 4

$$V = \int_2^4 \left[\int_1^3 \left(x^2 + y^2 \right) dx \right] dy$$

$$\int_1^3 \left(x^2 + y^2 \right) dx = \left[\frac{1}{3} x^3 + xy^2 \right]_1^3 = \left[\frac{1}{3} (3)^3 + (3) y^2 \right] - \left[\frac{1}{3} (1)^3 + (1) y^2 \right]$$

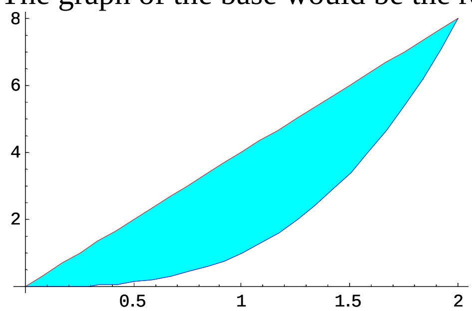
$$= 9 + 3y^2 - \frac{1}{3} - y^2 = \frac{26}{3} + 2y^2$$

$$\int_2^4 \left[\int_1^3 \left(x^2 + y^2 \right) dx \right] dy = \int_2^4 \left[\frac{26}{3} + 2y^2 \right] dy = \left[\frac{26}{3} y + \frac{2}{3} y^3 \right]_2^4$$

$$= \left[\frac{26}{3}(4) + \frac{2}{3}(4)^3 \right] - \left[\frac{26}{3}(2) + \frac{2}{3}(2)^3 \right] = \frac{104}{3} + \frac{128}{3} - \frac{52}{3} - \frac{16}{3} = \frac{164}{3}$$

Example: Find the volume of the solid under the surface f(x, y) = 3 and over the region R in the first quadrant bounded by the curves y = 4x and $y = x^3$.

The graph of the base would be the region where y = 4x is red and $y = x^3$ is blue



$$R = \left\{ (x, y) \mid x^3 \le y \le 4x \right\}$$

$$V = \iint_R f(x, y)\, dx\, dy = \int_0^2 \left[\int_{x^3}^{4x} 3\, dy \right] dx$$

$$\int_{x^3}^{4x} 3\, dy = [3y]_{x^3}^{4x} = \left[12x - 3x^3 \right]$$

$$V = \int_0^2 \left[\int_{x^3}^{4x} 3\, dy \right] dx = \int_0^2 \left[12x - 3x^3 \right] dx = \left[6x^2 - \frac{3}{4} x^4 \right]_0^2$$

$$= \left[6(2)^2 - \frac{3}{4}(2)^4 \right] - \left[6(0)^2 - \frac{3}{4}(0)^4 \right] = 24 - 12 = 12$$