Section11.2

Probability Density Function

A function f(x) defined on an open interval (a, b) is a probability density function (pdf) if it satisfies the following conditions:

1. For all $x \in (a,b),\ f(x) \geq 0$.
2. $\int_a^b f(x)\,dx = 1$

Example: Show that the function f(x) = 1/(2x), with $1 < x < e^2$ is a probability density function.

1. If $1 < x < e^2$, then f(x) > 0
2. Look at the integral

$$\int_1^{e^2} \frac{1}{2x}\,dx = \int_1^{e^2} \frac{1}{2}\cdot\frac{1}{x}\,dx = \left[\frac{1}{2}\ln x\right]_1^{e^2} = \left[\frac{1}{2}\ln e^2\right] - \left[\frac{1}{2}\ln 1\right] = \frac{1}{2}(2) - \frac{1}{2}(0) = 1$$

So this is a probability density function

Probability of a Continuous Random Variable:

Let f(x) defined on (a, b) be a probability density function. Then the probability that a random variable X is in the interval (c, d) where the interval (c, d) is contained in the interval (a, b) is

$$P(c < X < d) = \int_c^d f(x)\,dx$$

Example: For the probability density function above, find P(1.5 < X < 3).

$$\int_{1.5}^{3} \frac{1}{2x}\,dx = \left[\frac{1}{2}\ln x\right]_{1.5}^{3} = \left[\frac{1}{2}\ln 3\right] - \left[\frac{1}{2}\ln 1.5\right]$$

$$= \frac{1}{2}[\ln 3 - \ln 1.5] = \frac{1}{2}\ln\left[\frac{3}{1.5}\right] = \frac{1}{2}\ln 2 \approx \frac{1}{2}(0.69) = 0.35$$

Uniform Probability Density Function: The Uniform probability density function on the interval [a, b] is the function f(x) = 1/(b – a).

Example: On the interval [1, 4], the uniform probability density function would be f(x) = 1/3
So, the probability that we get an outcome between 1.5 and 2 would be

$$\int_{1.5}^{2} \frac{1}{3}\,dx = \left[\frac{1}{3}x\right]_{1.5}^{2} = \left[\frac{1}{3}(2)\right] - \left[\frac{1}{3}(1.5)\right] = \frac{2}{3} - \frac{1}{3}\left(\frac{3}{2}\right) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

NOTE:

For the uniform pdf on [a, b]. So $f(x) = \dfrac{1}{b-a}$.

If $a \leq c < d \leq b,\ P(c < X < d) = \int_c^d \frac{1}{b-a}\,dx = \left[\frac{1}{b-a}x\right]_c^d = \frac{d}{b-a} - \frac{c}{b-a} = \frac{d-c}{b-a}$

Exponential Probability Density Function: The probability density function $f(x) = ke^{-kx}$, $x \geq 0$, is called the exponential probability density function.

Note: $\int_0^\infty k\,e^{-kx}\,dx = 1$

Example: consider the exponential probability density function

$$f(x) = \frac{1}{10}e^{-0.1x}, x \geq 0.$$

$$P(0 < X < 2) = \int_0^2 \frac{1}{10}e^{-0.1x}dx = \left[\frac{1}{10}\left(\frac{1}{-0.1}e^{-0.1}x\right)\right]_0^2$$

$$= \left[-e^{-0.1x}\right]_0^2 = \left[-e^{-0.1(2)}\right] - \left[-e^{-0.1(0)}\right] = -e^{-0.2} + 1 = 1 - e^{-0.2} = 1 - \frac{1}{e^{0.2}}$$

$$P(X > 2) = 1 - P(0 < X < 2) = 1 - \left(1 - \frac{1}{e^{0.2}}\right) = e^{-0.2}$$

$$NOTE: \int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$$