Section 11.2

Probability Density Function

A function f(x) defined on an open interval (a, b) is a probability density function (pdf) if it satisfies the following conditions:

1. For all $x \in (a,b)$, $f(x) \geq 0$.
2. $\int_a^b f(x)\,dx = 1$

Example: Show that the function f(x) = 1/(2x), with $1 < x < e^2$ is a probability density function.

Property 1: If $1 < x < e^2$, then f(x) > 0

Property 2

$$\int_1^{e^2} \frac{1}{2x}\,dx = \int_1^{e^2} \frac{1}{2}\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}\ln x\right]_1^{e^2} = \left[\frac{1}{2}\ln e^2\right] - \left[\frac{1}{2}\ln 1\right] = \frac{1}{2}(2) - \frac{1}{2}(0) = 1$$

So this is a probability density function

Probability of a Continuous Random Variable:

Let f(x) defined on (a, b) be a probability density function. Then the probability that a random variable X is in the interval (c, d) where the interval (c, d) is contained in the interval (a, b) is ($a \leq c < d \leq b$)

$$P(c < X < d) = \int_c^d f(x)\,dx$$

Example: For the probability density function above, find P(1.5 < X < 3).

$$P(1.5 < X < 3) = \int_{1.5}^{3} \frac{1}{2x}\,dx = \left[\frac{1}{2}\ln x\right]_{1.5}^{3} = \left[\frac{1}{2}\ln 3\right] - \left[\frac{1}{2}\ln 1.5\right]$$

$$= \frac{1}{2}[\ln 3 - \ln 1.5] = \frac{1}{2}\ln\left(\frac{3}{1.5}\right) = \frac{1}{2}\ln 2$$

Uniform Probability Density Function: The Uniform probability density function on the interval [a, b] is the function f(x) = 1/(b – a).

NOTE:

For the uniform pdf on [a, b], $f(x) = \dfrac{1}{b-a}$

$a \leq c < d \leq b$

$$P(c < x < d) = \int_c^d \frac{1}{b-a}\,dx = \left[\frac{1}{b-a}x\right]_c^d = \frac{d}{b-a} - \frac{c}{b-a} = \frac{d-c}{b-a}$$

Exponential Probability Density Function: The probability density function $f(x) = ke^{-kx}$, $x \geq 0$, is called the exponential probability density function.

Note: $\int_0^{\infty} k\,e^{-kx}\,dx = 1$

Example: consider the exponential probability density function

$$f(x) = \frac{1}{10} e^{-0.1x}, x \geq 0.$$

$$P(0 < X < 3) = \int_0^3 \frac{1}{10} e^{-0.1x} dx = \left[\left(\frac{1}{10} \right) \left(\frac{1}{-0.1} e^{-0.1x} \right) \right]_0^3$$

$$= \left[-e^{-0.1(3)} + e^0 \right] = 1 - e^{-0.3}$$

$$P\left(X > 3\right) = 1 - P(0 < X < 3)$$