Lecture Sec. 11.3

The Expected Value or Mean of a random variable X with a continuous probability density function f(x) on the interval (a, b) is

$$E(X) = \mu = \int_a^b x\, f(x)\, dx.$$

$$a < \mu < b$$

Example: f(x) = $x^3/20$ on [1, 3]. Find the expected value or mean.

$$\mu = \int_1^3 x\left(\frac{x^3}{20}\right) dx = \frac{1}{20}\int_1^3 x^4\, dx = \left[\frac{1}{20}\left(\frac{1}{5}\right)x^5\right]_1^3$$

$$= \frac{1}{100}\left(3^5\right) - \frac{1}{100}\left(1^5\right) = \frac{243}{100} - \frac{1}{100} = \frac{242}{100} = 2.42$$

The Variance of a random variable X with a continuous probability density function f(x) on the interval (a, b) is

$$Var(X) = \int_a^b (x-\mu)^2\, f(x)\, dx = \int_a^b x^2 f(x)\, dx - \mu^2$$

$$= \left[\int_a^b x^2 f(x)\, dx\right] - \mu^2$$

The Standard Deviation of a random variable X with a continuous probability density function f(x) on the interval (a, b) is

$$\sigma = \sqrt{Var(X)}.$$

Example continued: find the variance and standard deviation.

f(x) = $x^3/20$ on [1, 3]. $\mu = 2.42$

$$\int_1^3 x^2\left(\frac{x^3}{20}\right) dx = \frac{1}{20}\int_1^3 x^5\, dx = \left[\frac{1}{20}\left(\frac{1}{6}\right)x^6\right]_1^3 = \frac{1}{120}3^6 - \frac{1}{120}1^6$$

$$= \frac{729-1}{120} = \frac{728}{120} = \frac{91}{15} \approx 6.067$$

$$Var(X) = 6.067 - (2.42)^2 \approx 0.2103$$

$$\sigma = \sqrt{Var(x)} = \sqrt{0.2103} = 0.459$$

Example: Let f(x) = 1 / (b – a) on [a, b]. (Uniform pdf)

$$\mu = \int_a^b x\left(\frac{1}{b-a}\right)dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{(b-a)(b+a)}{2(b-a)} = \frac{1}{2}(a+b)$$

$$Var = \int_a^b x^2\left(\frac{1}{b-a}\right)dx - \left[\frac{1}{2}(a+b)\right]^2 = \frac{1}{3}\left(b^2 - ab + a^2\right) - \frac{1}{4}\left(a^2 + 2ab + b^2\right)$$

$$= \frac{1}{12}\left[4\left(b^2 + ab + a^2\right) - 3(a^2 + 2ab + b^2)\right] = \frac{1}{12}\left(b^2 - 2ab + a^2\right) = \frac{1}{12}(b-a)^2$$

$$\int_a^b x^2\left(\frac{1}{b-a}\right)dx = \left(\frac{1}{b-a}\right)\int_a^b x^2 dx = \left(\frac{1}{b-a}\right)\left[\left(\frac{1}{3}\right)x^3\right]_a^b = \frac{1}{3(b-a)}(b^3 - a^3)$$

$$= \frac{1}{3(b-a)}(b-a)(b^2 - ab + a^2) = \frac{b^2 - ab + a^2}{3}$$

$$\sigma = \sqrt{\frac{1}{12}(b-a)^2} = \sqrt{\frac{1}{12}}(b-a) = \frac{1}{2\sqrt{3}}(b-a)$$

Note: For the exponential probability density function f(x) = ke^{-kx}, k > 0, on [0, ∞), μ = 1/k and σ = 1/k.

Example: Suppose that we have determined through random sampling that the lifetime of an electronic part has an exponential probability density function with mean of 100 hours.

$$\frac{1}{k} = 100$$

$$k = \frac{1}{100}$$

$$f(x) = \frac{1}{100}e^{-\frac{1}{100}x}$$

The Median of a random variable X with a continuous probability density function f(x) on the interval (a, b) is that number x_m such that

$$\int_a^{x_m} f(x)\,dx = \frac{1}{2}.$$