

Lecture Sec. 11.3

**The Expected Value or Mean** of a random variable  $X$  with a continuous probability density function  $f(x)$  on the interval  $(a, b)$  is

$$E(x) = \mu = \int_a^b x f(x) dx.$$

**Example:**  $f(x) = x^3/20$  on  $[1, 3]$ . Find the expected value or mean.

$$\mu = \int_1^3 x \left( \frac{x^3}{20} \right) dx = \frac{1}{20} \int_1^3 x^4 dx = \frac{1}{20} \left[ \frac{1}{5} x^5 \right]_1^3 = \frac{1}{100} 3^5 - \frac{1}{100} 1^5 = \frac{243}{100} - \frac{1}{100} = \frac{242}{100} = 2.42$$

**The Variance** of a random variable  $X$  with a continuous probability density function  $f(x)$  on the interval  $(a, b)$  is

$$\begin{aligned} \text{Var}(X) &= \int_a^b (x - \mu)^2 f(x) dx = \int_a^b x^2 f(x) dx - \mu^2 \\ &= \left[ \int_a^b x^2 f(x) dx \right] - \mu^2 \end{aligned}$$

**The Standard Deviation** of a random variable  $X$  with a continuous probability density function  $f(x)$  on the interval  $(a, b)$  is

$$\sigma = \sqrt{\text{Var}(X)}.$$

**Example continued:** find the variance and standard deviation.

$$\text{Var}(X) = \int_1^3 (x - 2.42)^2 \frac{x^3}{20} dx = \int_1^3 x^2 \frac{x^3}{20} dx - (2.42)^2$$

$$\int_1^3 \frac{x^5}{20} dx = \left[ \frac{1}{20} \left( \frac{1}{6} \right) x^6 \right]_1^3 = \left[ \frac{1}{120} 3^6 \right] - \left[ \frac{1}{120} 1^6 \right] = \frac{729}{120} - \frac{1}{120} = \frac{728}{120} = \frac{91}{15}$$

$$\text{Var}(X) = \frac{91}{15} - (2.42)^2 \approx 0.2103$$

$$\sigma = \sqrt{\text{Var}(x)} = \sqrt{0.2103} = 0.459$$

**Example:** Let  $f(x) = 1 / (b - a)$  on  $[a, b]$ . Uniform probability density function

$$\begin{aligned} \mu &= \int_a^b x \left( \frac{1}{b-a} \right) dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left[ \frac{1}{2} x^2 \right]_a^b = \frac{1}{b-a} \left[ \frac{1}{2} b^2 - \frac{1}{2} a^2 \right] \\ &= \frac{1}{2(b-a)} [b^2 - a^2] = \frac{1}{2(b-a)} [(b-a)(b+a)] = \frac{a+b}{2} \end{aligned}$$

**Note:** For the exponential probability density function  $f(x) = ke^{-kx}$ ,  $k > 0$ , on  $[0, \infty)$ ,  $\mu = 1/k$  and  $\sigma = 1/k$ .

**Example:** Suppose that we have determined through random sampling that the lifetime of an electronic part has an exponential probability density function with mean of 100 hours.

$$\mu = 100 = \frac{1}{k}$$

$$k = \frac{1}{100}$$

$$f(x) = \frac{1}{100} e^{-\frac{1}{100}x}, x \geq 0$$

$$\begin{aligned} P(0 < X < 50) &= \int_0^{50} \frac{1}{100} e^{-\frac{1}{100}x} dx = \frac{1}{100} \int_0^{50} e^{-\frac{1}{100}x} dx = \left[ \frac{1}{100} \left[ \frac{1}{-\frac{1}{100}} \right] e^{-\frac{1}{100}x} \right]_0^{50} \\ &= \left[ -e^{-\frac{1}{100}x} \right]_0^{50} = \left[ -e^{-\frac{1}{100}(50)} \right] - \left[ -e^{-\frac{1}{100}(0)} \right] = -e^{-\frac{1}{2}} + 1 \approx 0.39 \end{aligned}$$

**The Median** of a random variable X with a continuous probability density function f(x) on the interval (a, b) is that number  $x_m$  such that

$$\int_a^{x_m} f(x) dx = \frac{1}{2}.$$

**Example:** From the electronic parts, what is the median lifetime?

Solve  $\int_0^{x_m} \frac{1}{100} e^{-\frac{1}{100}x} dx = \frac{1}{2}$  for  $x_m$ .

$$\int_0^{x_m} \frac{1}{100} e^{-\frac{1}{100}x} dx = \left[ -e^{-\frac{1}{100}x} \right]_0^{x_m} = \left[ -e^{-\frac{1}{100}x_m} \right] - \left[ -e^{-\frac{1}{100}(0)} \right] = \left[ -e^{-\frac{1}{100}x_m} \right] + 1 = \frac{1}{2}$$

$$e^{-\frac{1}{100}x_m} = \frac{1}{2}$$

$$x = 100 \ln 2 = 69.3$$

median life of electronic part is about 69 hours