Section 8.4

Problem: Find the minimum value(s) for the function $f(x, y) = xy^2$ in the first quadrant subject to the constraint $g(x, y) = x^2 + y^2 = 8$.

General Statement: Optimize a function f(x, y) given a constraint that g(x, y) = 0
Method of Lagrange multipliers

Step 1: Define a new function F(x, y, L) f(x, y) + Lg(x, y)
$F(x, y, L) = xy^2 + L(x^2 + y^2 - 8)$

Step 2: Solve $F_x = 0$, $F_y = 0$, and $F_L = 0$

a. Find each of the partial derivatives and set = 0

$F_x = y^2 + 2xL = 0$
$F_y = 2xy + 2yL = 0$
$F_L = x^2 + y^2 - 8 = 0$

b. Solve each of the top two equations for L

$$y^2 = -2xL \Rightarrow L = \frac{y^2}{-2x}$$

$$2xy = -2yL \Rightarrow L = \frac{2xy}{-2y} = -x$$

c. Set the two things equal and solve for one variable

$$\frac{y^2}{-2x} = -x$$

$$y^2 = 2x^2$$

d. and plug that into the equation $F_L = 0$

$x^2 + y^2 - 8 = 0$
$x^2 + 2x^2 - 8 = 0$
$3x^2 = 8$
$x^2 = 8/3$

Step 3: Find the points:

$$x^2 = \frac{8}{3};\ x = \pm\sqrt{\frac{8}{3}} = \pm\sqrt{\frac{24}{9}} = \pm\frac{\sqrt{24}}{3} = \pm\frac{2\sqrt{6}}{3}$$

$$y^2 = 2x^2 = \frac{16}{3};\ y = \pm\sqrt{\frac{16}{3}} = \pm\frac{4}{\sqrt{3}} = \pm\frac{4\sqrt{3}}{3}$$

Step 4: Give the answer

We have 4 points: $\left(\frac{2\sqrt{6}}{3}, \frac{4\sqrt{3}}{3}\right), \left(\frac{2\sqrt{6}}{3}, -\frac{4\sqrt{3}}{3}\right), \left(-\frac{2\sqrt{6}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{2\sqrt{6}}{3}, -\frac{4\sqrt{3}}{3}\right)$

We need to plug these into the function $f(x, y) = xy^2$

So the minimum will be the 2 where x is negative; we would have a maximum if x were positive.