NAME:

MATH 242 FINAL PROBLEM SET, Due 4pm Friday May 9, 2008

Please return to Professor Sands's mailbox
Use no resources other than your Rosenlicht textbook
Be sure to state what results or definitions you use.

1. Rosenlicht, p. 246, # 18

2. (Volume of Revolution) Suppose that $g : [a, b] \to \mathbb{R}$ is a positive-valued continuous function which we graph in the xz-plane by considering $x = g(z)$. We let A be the region between the graph and the z-axis: $A = \{(x, z) \in \mathbb{R}^2 : 0 \leq x \leq g(z), z \in [a, b]\}$. Define C to be the set obtained by revolving A around the z-axis in xyz-space: $C = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)^{1/2} \leq g(z), z \in [a, b]\}$. Show that

$$\mathrm{vol}(C) = \pi \int_a^b [g(z)]^2 \, dz$$

   Use the cylindrical coordinates map $\phi(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$ and observe that C is the image under ϕ of the set $B = \{(r, \theta, z) \in \mathbb{R}^3 : 0 \leq r \leq g(z), \theta \in [0, 2\pi], z \in [a, b]\}$. Be sure to mention how you use the change of variables theorem and Fubini's theorem.

3. Let A and B be n by n matrices with real entries. Let $||A||$ denote the usual matrix norm of A.
   a. Prove that $||AB|| \leq ||A|| \cdot ||B||$
   b. For any integer $m > 0$, show that $||A^m|| \leq ||A||^m$.
   c. Show that each entry of the matrix

$$F(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

   converges to a real-valued function on $\mathbb{R}$ which is infinitely differentiable (i. e. has derivatives of all orders). You may use the facts that $||A + B|| \leq ||A|| + ||B||$, $||cA|| = |c| \cdot ||A||$ for any real number c, and $||B||$ is greater than or equal to the absolute value of any entry of B.
   d. Show that if you differentiate each entry, you get $F'(t) = AF(t)$. This is used to solve systems of differential equations.

4. Suppose that U is an open subset of $\mathbb{R}^n$, and $F : U \to \mathbb{R}^n$ is a function with continuous first partial derivatives on U. Also suppose that the Jacobian determinant $J_F(a) \neq 0$ for each $a \in U$.
   a. Show that each point $a \in U$ is contained in an open set $U_a \subset U$ such that $F(U_a)$ is also an open set.
   b. Show that if V is any open set in U, then $F(V)$ is open. This is part of the open mapping theorem.