Fourier Series Representations

Introduction. Before we discuss the technical aspects of Fourier series representations, it might be well to discuss the broader question of why they are needed. We'll begin by looking at the earliest recognizable application of such representations, which was in the problem of mathematically modeling the vibrations of a stretched string. Afterwards, we'll discuss the problem of how to analyze a periodic function by separating it into its components frequencies, each with its own amplitude and phase. Then we'll discuss the opposite problem of synthesizing the function from its Fourier series.

1. The Vibrating String Problem

Isaac Newton died in 1727. By the time of his death, his laws of mechanics, which he stated in a complicated geometric form, had been reformulated by the Continental mathematicians into a form that we can recognize today. (If you read Newton's original work, you will not recognize anything that you have seen in a physics textbook.) In particular, by the mid-eighteenth century, the mathematicians Leonhard Euler, Daniel Bernoulli, and Jean Le Rond d'Alembert were attempting the mathematical analysis of vibrating strings and membranes. We are going to look at the simplest case of the vibrating string, as modeled and studied in the 1740s by d'Alembert and Bernoulli. This model neglects friction, air resistance, and gravitation and makes the assumption that the string is clamped at both ends, which are located at $(0, 0)$ and $(\ell, 0)$, where ℓ is the natural length of the string.

1. The mathematical model. The string is assumed to have been stretched into some arbitrary shape represented by the graph of a function $y = f(x)$, then released from rest at time $t = 0$, so that at any subsequent time t the shape of the string will be the graph of a function $y = u(x, t)$, as shown in Fig. 1 below. The differential equation that describes the motion for positive values of t is given in linear approximation by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,, \tag{1}$$

where the constant c must have the dimensions of a velocity in order for the equation to make physical sense. That is, the left-hand side contains the square of time in the denominator and the right-hand side contains the square of distance in its denominator. In order to make the equation balance physically, c must be a velocity. This is a *homogeneous* equation, in the sense that if u and v satisfy it, so does $au + bv$ for any constants a and b. This means that particular solutions can be scaled and superimposed to produce new ones

Besides the equation itself, our assumptions impose two *homogeneous* boundary conditions on the solution u, namely

$$u(0, t) = 0 = u(\ell, t) \tag{2}$$

for all times t. These equations express the fact that the string is clamped at both ends. Again, homogeneity means that when solutions of Eq. (1) satisfying these equations are scaled and/or superimposed, the new solutions will also satisfy Eq. (2).

In addition, we have assumed the string is released from rest, that is, its initial velocity is zero, and that leads to a *homogeneous* initial condition on u, namely

$$\frac{\partial u}{\partial t} = 0 \tag{3}$$

when $t = 0$, for all values of x, $x \in [0, \ell]$.

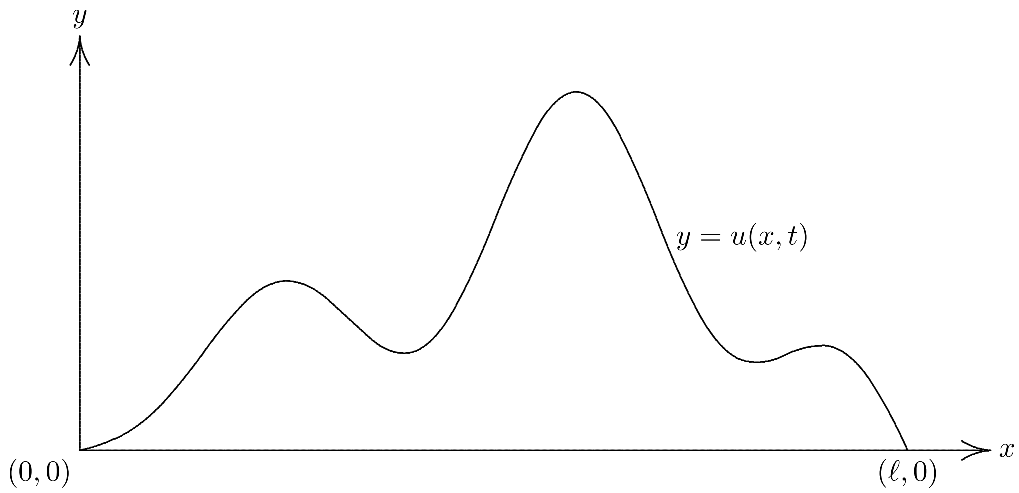

Fig. 1. Graph of the solution $y = u(x,t)$ at an arbitrary time t

Finally, we have imposed the *non-homogeneous* initial condition

$$u(x,0) = f(x)\,. \tag{4}$$

Since this condition is non-homogeneous, we cannot add or scale solutions that satisfy it. It is this condition that really chooses a unique solution out of the infinitely many solutions that will satisfy the other conditions.

Exercise 1. It appears that we have imposed too many conditions to be satisfied if $f(x)$ is completely arbitrary, since (2) and (4) together imply that $f(0) = 0 = f(\ell)$. Show that this restriction on $f(x)$ can be removed by replacing $f(x)$ with $g(x) = f(x) - f(0) + \frac{x}{\ell}\big(f(0) - f(\ell)\big)$ in condition (4). (*Hint*: Since $g(x)$ does satisfy this additional condition, if we can solve the problem (1)–(4) with $g(x)$ in place of x to get $u(x,t)$, then $v(x,t) = u(x,t) + f(0) + \frac{x}{\ell}\big(f(\ell) - f(0)\big)$ will satisfy Eq. (1), the homogeneous initial condition (3), and the non-homogeneous conditions $v(0,t) = f(0)$, $v(\ell,t) = f(\ell)$, and $v(x,0) = f(x)$.

2. D'Alembert's solution. The problem we have just posed turns out to have a very simple and elegant condition if the initial shape of the string is smooth enough. This solution is due to d'Alembert. The solution is found by extending the graph of the initial function outside the interval $[0,\ell]$ to the entire real line as an *odd function of period* 2ℓ, as shown in Fig. 2 below. After this is done, the solution can be written in closed form as

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}\,. \tag{5}$$

Exercise 2. Verify that $u(x,t)$ really does satisfy all of the equations (1)–(4) if $f(x)$ is an odd function of period 2ℓ. (Only Eq. (2) is a little bit tricky. The fact that $u(0,t) \equiv 0$ follows because f is an odd function. The equation $u(\ell,t) \equiv 0$ follows if you write Eq. (5) in the form

$$u(x,t) = \frac{f(ct+x) - f(ct-x)}{2}$$

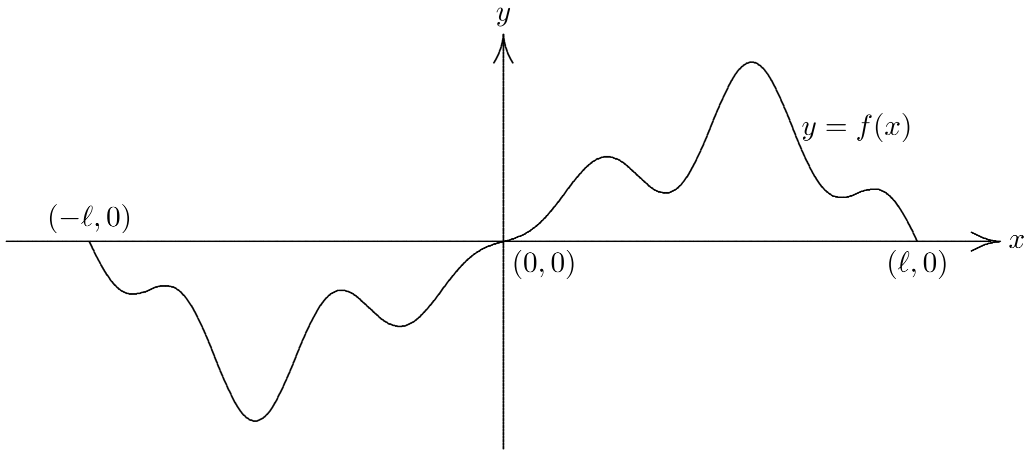

Fig. 2. Extension of $f(x)$ as an odd function

and use the fact that f has period 2ℓ.)

3. Interpretation of d'Alembert's solution. The neat thing about d'Alembert's solution is its elegant geometric interpretation. If you interpret the odd periodic function $f(x)$ as a wave, then the function $f(x+ct)$ is that wave moving left with velocity c, and the function $f(x-ct)$ is the same wave moving right with velocity c. The solution is then the average of these two wave motions. Could we possibly want anything more?

Actually, we can, or at least other people did. We have been working formally here. We'd really like to model cases where the initial position of the string is not so smooth. It may have kinks in it, so that $f''(x)$ doesn't exist at some points. In particular, Bernoulli wanted to consider the case of the *plucked string* shown in Fig. 2 below. For this case, the initial function $f(x)$ was given by

$$f(x) = h\left[\left|\frac{\ell}{2} - x\right| - \frac{\ell}{2}\right], \tag{6}$$

where h is some positive scaling factor. The graph of this function is shown in Fig. 3.

In that case, although it's a seemingly small matter that most physicists would gloss over, mathematicians have to worry about what happens. But at least, d'Alembert's solution is probably going to be a guide to our work.

4. Daniel Bernoulli's solution. Not knowing what kind of function $u(x,t)$ might satisfy Eqs. (1)–(4) in general, Daniel Benoulli adopted the hopeful-search technique, looking only for solutions of a special form, where he hoped to compute them directly from the equation. He was the first to use what we now call *separation of variables*, assuming that $u(x,t)$ has the form

$$u(x,t) = X(x)T(t)\,, \tag{7}$$

where $X(x)$ and $T(t)$ were smooth functions of one variable. Our boundary conditions impose the requirement that $X(0) = 0 = X(\ell)$, and our homogeneous initial condition means that $T'(0) = 0$.

If you work formally with Eq. (1), you find that it becomes

$$X(x)T''(t) = c^2 X''(x)T(t)\,,$$

which can be rewritten as

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)}\,. \tag{8}$$

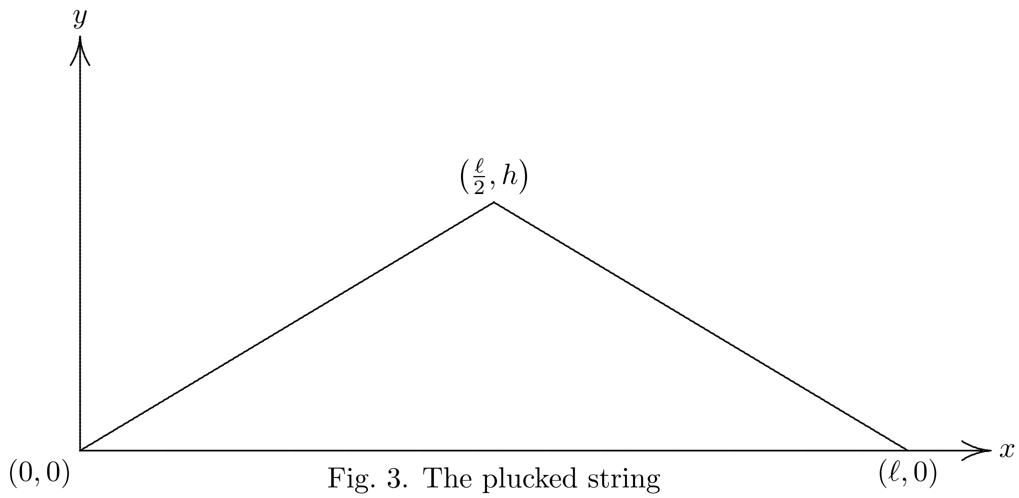

Fig. 3. The plucked string

Now the left-hand side of Eq. (8) is independent of x, and the right-hand side is independent of t. That means, since the two sides are equal, that both sides must be independent of both variables, hence they must be equal to a constant λ. We thus get the equations

$$T''(t) - \lambda T(t) = 0\,; \quad X''(x) - \frac{\lambda}{c^2} X(x) = 0\,. \tag{9}$$

Given that t represents time and x represents distance, λ must have the physical dimension time^{-2} in order for these equations to make physical sense.

If $\lambda > 0$, the general solution of these equations is known. It is

$$T(t) = \alpha e^{\sqrt{\lambda} t} + \beta e^{-\sqrt{\lambda} t}\,,$$
$$X(x) = \gamma e^{\frac{\sqrt{\lambda}}{c} x} + \delta e^{-\frac{\sqrt{\lambda}}{c} x}\,,$$

for arbitrary constants α, β, γ, and δ. But then our boundary conditions $X(0) = 0 = X(\ell)$ imply (by direct solution of the equations) that $\gamma = 0 = \delta$, and that makes this solution *identically zero.* Thus, although it is a solution, it is certainly not of any interest to us.

Exercise 3. Work through the case $\lambda = 0$, and show that in this case also we get $X(x) \equiv 0$, so that this possibility must also be rejected.

We have now reached the conclusion that λ must be negative, say $\lambda = -\mu$, where $\mu > 0$. (Once again, μ must have the physical dimension time^{-2}.) In that case also, the general solution of the system (9) is known. It is

$$T(t) = \alpha \cos(\sqrt{\mu} t) + \beta \sin(\sqrt{\mu} t)\,,$$
$$X(x) = \gamma \cos\big(\frac{\sqrt{\mu}}{c} x\big) + \delta \sin\big(\frac{\sqrt{\mu}}{c} x\big)\,.$$

This time, the equations $X(0) = 0 = X(\ell)$ imply something more interesting. They *usually* entail $X(x) \equiv 0$. Obviously the equation $X(0) = 0$ will imply that $\gamma = 0$, so that $X(x) = \delta \sin\big(\frac{\sqrt{\mu}}{c} x\big)$.

But we can then assume $\delta \neq 0$, and hence get a non-trivial $X(x)$, provided there is some integer n such that

$$\frac{\sqrt{\mu}}{c}\ell = n\pi\,.$$

That is,

$$\sqrt{\mu} = \frac{n\pi c}{\ell}\,.$$

Obviously, $n = 0$ is again not of interest, but n may be any positive integer. There would be no point in taking n negative, since doing so would merely replace $X(x)$ by $X(-x)$, which is just $-X(x)$.

With these possible values of n, our homogeneous initial condition $T'(0) = 0$ implies that $\beta = 0$, and so we are finally provided with a whole set of solutions

$$u_n(x,t) = c_n \cos\left(\frac{n\pi c}{\ell}t\right) \sin\left(\frac{n\pi}{\ell}x\right),$$

where we have condensed the product $\alpha\delta$ into a single arbitrary constant c_n. Because the solutions can be superimposed, this means we can satisfy the equation and all three homogeneous boundary conditions by any finite sum

$$u(x,t) = \sum_{n=1}^{N} c_n \cos\left(\frac{n\pi c}{\ell}t\right) \sin\left(\frac{n\pi}{\ell}x\right). \tag{10}$$

By Eq. (10), we find that Eq. (4) says

$$f(x) = u(x,0) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi}{\ell}x\right). \tag{11}$$

It follows that superimposing a finite number of solutions will enable us to solve the original problem only for certain special functions $f(x)$, namely those that are given by a trigonometric polynomial of the form (11).

5. Comparison of the solutions of d'Alembert and Bernoulli. We have just noted that a *finite* superposition of solutions obtained by separating variables can represent only strings whose initial configurations are the graphs of trigonometric polynomials. We now ask if the solution for these cases is the same one that d'Alembert found. The answer turns out to be easy.

Exercise 4. Use the identity

$$\cos a \sin b = \frac{1}{2}\big(\sin(b+a) + \sin(b-a)\big)$$

to rewrite Eq. (10) as

$$u(x,t) = \frac{\sum_{n=1}^{N} c_n \sin\left(\frac{n\pi}{\ell}(x+ct)\right) + \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi}{\ell}(x-ct)\right)}{2}\,. \tag{12}$$

Then compare (12) with (11) to conclude that Bernoulli's solution is the same as d'Alembert's for the given class of initial configurations $f(x)$.

2. Fourier Series

Although the initial configuration of the plucked string is not a trigonometric polynomial, Daniel Bernoulli thought it could be represented as an infinite series of trigonometric functions:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi}{\ell} x\Big) . \tag{13}$$

He claimed that any function $f(x)$ satisfying $f(0) = 0 = f(\ell)$ should be representable in this way. Euler, whose idea of a function was "something given by a formula," objected that only odd functions of period 2ℓ could be so represented.

Fortunately, we don't have to go into that controversy. We will instead consider the general problem of representing a function $f(x)$ by a trigonometric series. In order to streamline our formulas, we are going to assume henceforth that $\ell = \pi$. Thus, if we are given a function $f(x)$ on $[0, \pi]$ with $f(0) = 0 = f(\pi)$, we wish to choose c_n so that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) . \tag{14}$$

The basic question to be answered is whether this is possible at all, and if so, how the coefficients c_n need to be chosen.

1. Fourier coefficients. Assuming that such a representation exists, and further assuming that we can interchange the order of integration and summation (as we know we can do for *finite* sums), we find that for each positive integer m,

$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} c_n \int_0^{\pi} \sin(nx) \sin mx) \, dx .$$

Since $\sin a \sin b = \frac{1}{2}\big(\cos(a-b) - \cos(a+b)\big)$, we see that all the terms in the series with $n \neq m$ drop out:

$$\int_0^{\pi} \sin(nx)\sin(mx)\, dx = \frac{1}{2}\Big(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\Big)\Big|_0^{\pi} = 0 .$$

For $n = m$, we have $\sin^2 nx = \frac{1}{2} - \frac{1}{2}\cos(2nx)$, and so

$$\int_0^{\pi} \sin^2 nx \, dx = \frac{\pi}{2} .$$

The upshot of all this is that, if the representation is "good enough" so that we can integrate termwise, we *must* have

$$c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx . \tag{15}$$

Thus, Eq. (15) tells us how we must choose the coefficients c_n.

Standard intervals. Before we go on we need to get rid of one annoying restriction on the functions represented. Obviously, any function $f(x)$ that can be represented by a series (14) must satisfy $f(0) = 0 = f(\pi)$. In order to remove that restriction, we will allow ourselves to use the function $\cos nx$ as well as $\sin nx$. When we do that, we can double the length of the interval and represent

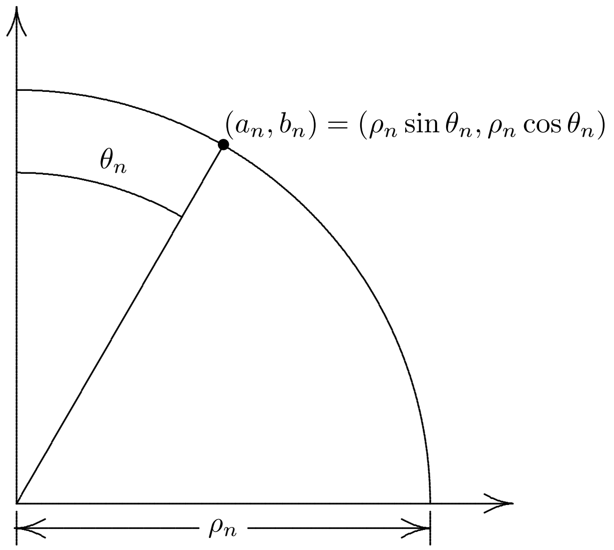

Fig. 4. Amplitude and phase

functions on $[-\pi, \pi]$ (or, as some authors prefer, on $[0, 2\pi]$). The Fourier series of a function $f(x)$ of period 2π is the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,, \tag{16}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt\,, \tag{17}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt\,. \tag{18}$$

This interval is chosen only for convenience, so that the frequencies can be written as pure integers. Any function $f(x)$ on any interval $[a, b]$ can be represented as the sum of a trigonometric series like (16), only using functions having period $b - a$, provided of course that $f(a) = f(b)$.

2. Amplitude and phase. Let $\rho_n = \sqrt{a_n^2 + b_n^2}$. The nonnegative number ρ_n is called the *amplitude* associated with the frequency n.

If the amplitude ρ_n is non-zero, then the point $\left(\frac{a_n}{\rho_n}, \frac{b_n}{\rho_n}\right)$ lies on the unit circle, and hence there exists an angle θ_n (measured clockwise from the positive y-axis), called the *phase* associated with the frequency n, such that

$$\frac{a_n}{\rho_n} = \sin\theta_n\,, \quad \frac{b_n}{\rho_n} = \cos\theta_n\,,$$

We then have

$$a_n \cos nx + b_n \sin nx = \rho_n(\cos nx \sin\theta_n + \sin nx \cos\theta_n) = \rho_n \sin(nx + \theta_n)\,, \tag{19}$$

as shown in Fig. 4.

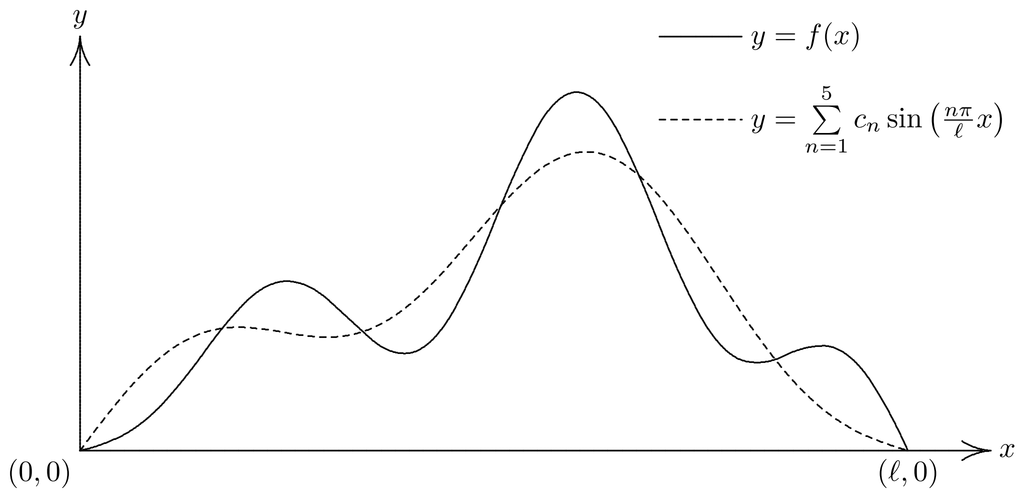

Fig. 5. Graph of a function $f(x)$ and the sum of the first five terms of its Fourier series

Exercise 5. The factor $\frac{1}{2}$ in front of a_0 is needed in order to make Eq. (17) valid for all n, including 0. We need not discuss b_0 at all, since it would obviously be zero. Prove that *if* the series (16) converges to $f(x)$ and termwise integration is allowed, then a_n and b_n *must* be given by formulas (17)–(18).

Now that we have answered the question of how to analyze a function with a trigonometric series (by showing how the coefficients are to be computed from the function, we are now ready to consider the opposite question: Does the series (16), with coefficients given by (17)–(18) converge and represent the function $f(x)$? A trigonometric series for which there exists a function $f(x)$ such that the coefficients a_n and b_n are given by the formulas (17)–(18) is called a *Fourier series.* Specifically, it is called the Fourier series of the function $f(x)$. *Not every trigonometric series is a Fourier series.* One that isn't, even though it converges at every point, is

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}\,.$$

(According to Parseval's Theorem, if a series $\sum_{n=1}^{\infty} c_n \sin nx$ converges to a square-integrable function, the sum of the squares of the coefficients must converge.)

3. Some examples of the convergence of Fourier series. We hope that adding up the terms of the Fourier series of a function will reproduce the function that generated the series. Just to get some empirical evidence that this hope is justified, we present a few examples here.

We begin with a more or less general function $f(x)$ (actually the function whose graph is shown in Fig. 1 above) and compare its graph with the graphs obtained by taking first five terms (Fig. 5), then six terms (Fig. 6) of the series. As you can see, with five terms the approximation doesn't look very good, but with six terms it looks much better. With seven or more terms, the approximation is so good that the graphs are virtually indistinguishable.

The plucked string presents a small challenge to the Fourier series representation because of the kink in its graph. (The functions we are using to represent it have no kinks. They are all

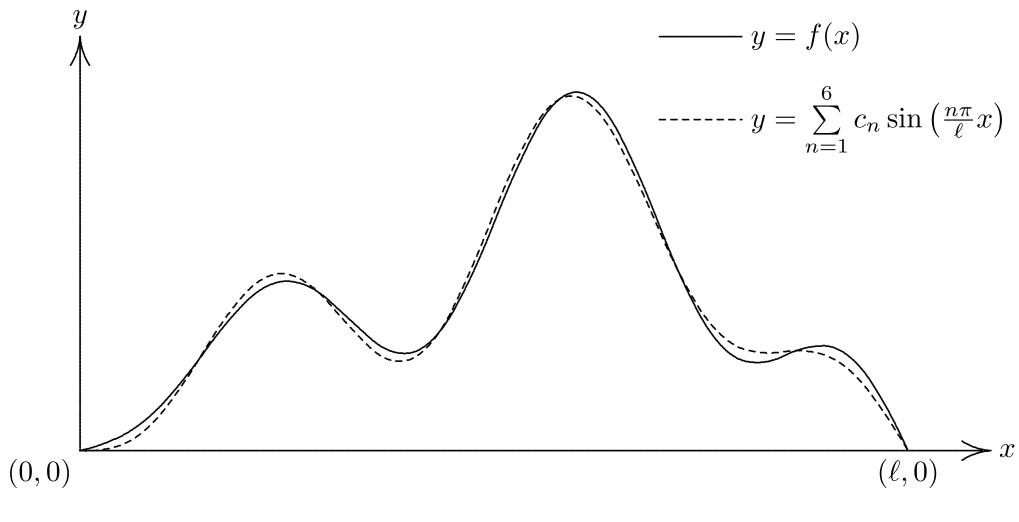

Fig. 6. Graph of a function $f(x)$ and the sum of the first six terms of its Fourier series

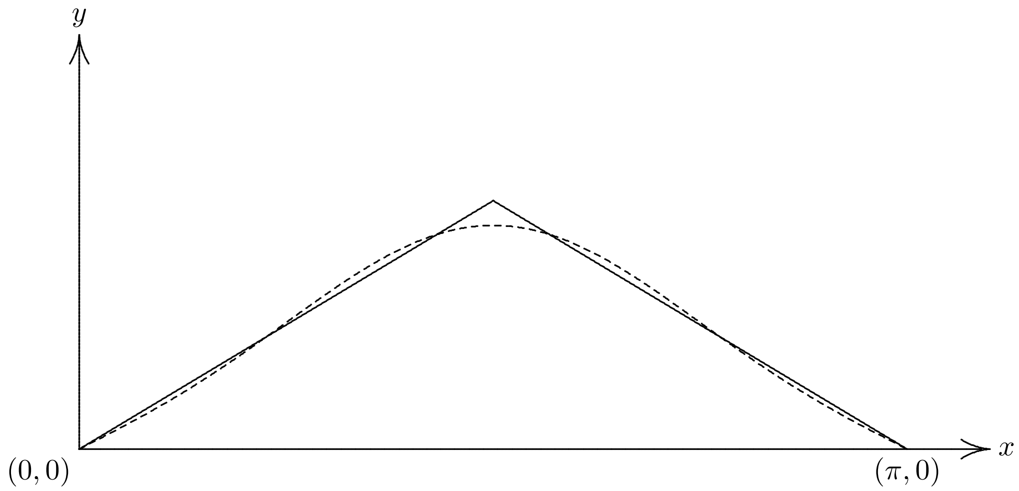

Fig. 7. Graph of the plucked string and the sum of the first three terms of its Fourier series

infinitely smooth.) However, this is not a serious challenge. The Fourier series of this function, if we take $\ell = \pi$, is

$$\frac{12}{5 \cdot 1\pi^2}\sin(x) + \frac{12}{5 \cdot 9\pi^2}\sin(3x) + \frac{12}{5 \cdot 25\pi^2}\sin(5x) + \cdots .$$

The graph of this function compared with the first three terms of this series (counting the second term as zero, since there is no term $\sin(2x)$) is shown in Fig. 7.

To get a really serious challenge to the representation, we need to consider a function that

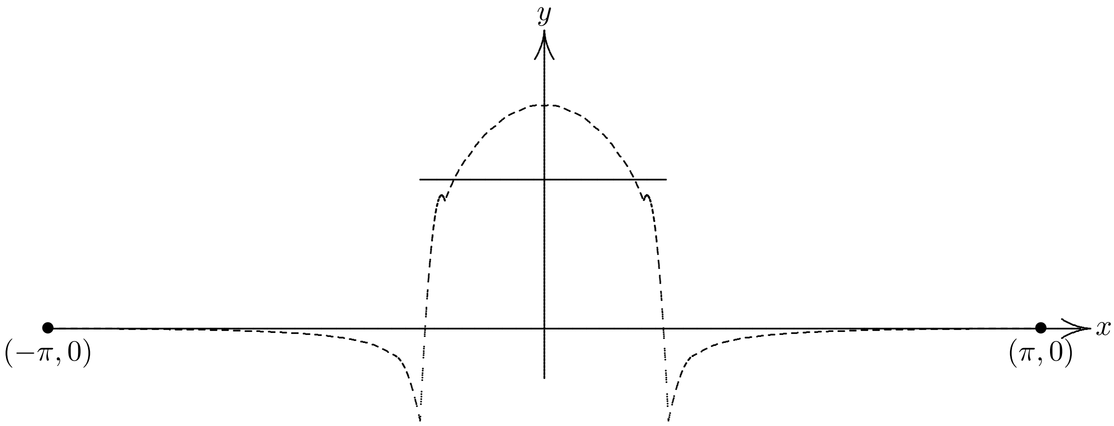

Fig. 8. Graph of a discontinuous function and a partial sum of its Fourier series

has discontinuities. Here's an example, which for convenience we represent as an even function of period 2π, on the interval $[-\pi, \pi]$. Since this function is even, we'll need only cosines to represent it. The function we choose is

$$f(x) = \begin{cases} \frac{3\pi}{10} & \text{if } |x| \le \pi/4\,, \\ \\ 0 & \text{if } \pi/4 < |x| \le \pi\,. \end{cases}$$

This function has a Fourier series whose initial terms are

$$\frac{3\pi}{40} + \frac{3}{5\sqrt{2}}\cos(x) + \frac{3}{10}\cos(2x) + \\ + \frac{3}{15\sqrt{2}}\cos(3x) - \frac{3}{25\sqrt{2}}\cos(5x) - \frac{3}{30}\cos(6x) - \frac{3}{35\sqrt{2}}\cos(7x) + \\ + \frac{3}{45\sqrt{2}}\cos(9x) + \frac{3}{50}\cos(10x) + \cdots\,.$$

The graph of $f(x)$ compared with the graph of the sum of the first 200 terms of its Fourier series is shown in Fig. 8, so that you can see just how difficult it is for this series to approximate $f(x)$. There is a noticeable tendency for the partial sums to "overshoot" the gap at each point of discontinuity. This overshoot is known as *Gibbs' phenomenon.*

Exercise 6. Take $x = 0$ in the Fourier series just written and verify that the sum really is $f(0)$, that is, 0.3π. To do that, you will need to know the formulas

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$
$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots\,.$$

These formulas can be proved in a standard way using power series. (The first one is commonly exhibited in calculus books.)

3. Convergence of Fourier Series

We are going to prove a very elementary criterion for convergence of a Fourier series. Before doing so, we need to establish two classical and easy results.

Theorem (Riemann–Lebesgue Lemma). *If $f(x)$ is an integrable function on $[-\pi, \pi]$, then its Fourier coefficients a_n and b_n tend to zero as $n \to \infty$.*

Proof: This is a matter of immediate computation if $f(x)$ is the characteristic function of an interval $[a, b]$ contained in $[0, \pi]$, since in that case, for example

$$b_n = \frac{\cos(na) - \cos(nb)}{\pi n}.$$

The same result then extends easily to "step functions," which are linear combinations of such characteristic functions. For general integrable functions $f(x)$, having Fourier coefficients given by (17)–(18), we proceed as follows. Let $\varepsilon > 0$ be given. Let $g(x)$ be a step function such that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|\, dx < \frac{\varepsilon}{2}.$$

(For Riemann-integrable functions $f(x)$, the existence of such a step function $g(x)$ is practically the definition of Riemann integrability. For more general integrals, such as the Lebesgue integral, the existence of $g(x)$ is a theorem, but not a very difficult one to prove.) Let the Fourier sine coefficients of $g(x)$ be

$$d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)\, \sin(nx)\, dx. \tag{20}$$

By what has already been shown, there exists an integer N such that $|d_n| < \frac{\varepsilon}{2}$ for all $n > N$. Then, if $n > N$, we have

$$\begin{aligned}
|b_n| &\le |d_n| + |b_n - d_n| \\
&< \frac{\varepsilon}{2} + \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \big(f(x) - g(x)\big)\, \sin(nx)\, dx \right| \\
&\le \frac{\varepsilon}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|\, dx \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The proof that $b_n \to 0$ is now complete, and of course the proof that $a_n \to 0$ is identical to it.

The technique just used to prove the Riemann–Lebesgue Lemma has many applications in analysis. The next two exercises are included to give you some practice in mastering the technique and applying it to produce a theorem.

Exercise 7. Let $f(x)$ be a function that is defined on the entire real line and integrable over each finite interval of the real line, and let $[a, b]$ be any fixed finite interval. (Any continuous function will do for $f(x)$, for example, and so will any step function). For each real number t, define a function $g_t(x)$ by $g_t(x) = f(x + t)$. Prove that the function $\varphi(t)$ defined by

$$\varphi(t) = \int_a^b g_t(x)\, dx$$

is a continuous function. In particular

$$\lim_{t \to 0} \varphi(t) = \varphi(0) = \int_a^b f(x)\, dx.$$

(*Hint*: First assume $f(x)$ is the characteristic function of an interval $[c,d]$, that is, $f(x) = \chi_{[c,d]}(x)$, where

$$\chi_{[c,d]}(x) = \begin{cases} 1 & \text{if } x \in [c,d]\,, \\ 0 & \text{if } x \notin [c,d]\,. \end{cases}$$

You'll have a number of messy cases to do, depending on how a, b, c, and d are situated, but this is the only mess in the proof. After that, the extension to step functions and then to integrable functions is done exactly as in the proof of the Riemann–Lebesgue Lemma.)

Exercise 8. Define a set E to be *measurable* if its characteristic function $\chi_E(x)$ is integrable over each finite interval $[a,b]$. For sets E actually contained in an interval $[a,b]$ we can define the *measure* of E to be

$$\mu(E) = \int_a^b \chi_E(x)\,dx\,.$$

(This definition is independent of the interval $[a,b]$ containing E.) If E is an interval, it is measurable, and its measure is its length, but there may be other measurable sets as well, such as a union of intervals. Let E be a set of positive measure. Prove that the set of differences $\{x-y : x,y \in E\}$ contains some interval $(-\delta,\delta)$ with $\delta > 0$. (*Hint*: Apply the technique of Exercise 6, this time to the function $\chi_E(x)\chi_E(x+t)$, to prove that the integral of this function with respect to x is a continuous function of t. Since the integral of this function equals the measure of E when $t = 0$, which is positive, there is some interval $(-\delta,\delta)$ for which the integral has a positive value if t lies in this interval. That means that if $|t| < \delta$, there must be at least one $x \in E$ such that $x+t$ also belongs to E, that is, $t = (x+t) - x$ is the difference of two points in E.)

1. The Dirichlet kernel. We next need an explicit formula for the partial sums of the Fourier series.

To get this formula, we observe that

$$\begin{aligned}\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big(\frac{1}{2} + \sum_{n=1}^{N}\big(\cos(nt)\cos(nx) + \sin(nt)\sin(nx)\big)\Big)\,dt\,. \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big(\frac{1}{2} + \sum_{n=0}^{N}\cos\big(n(x-t)\big)\Big)\,dt\end{aligned}$$

The expression we need to deal with is inside the large brackets in this last expression. It is called the *Dirichlet kernel* (evaluated at $x-t$ in this case). We need to get a better expression for this sum, which we denote D_N:

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\,.$$

Using the identity

$$\sin a \cos b = \frac{1}{2}\big(\sin(b+a) - \sin(b-a)\big)\,,$$

we get

$$\begin{aligned}\sin(x/2)D_N(x) = \frac{1}{2}\Big(\sin(x/2) &+ \big(\sin(3x/2) - \sin(x/2)\big) + \big(\sin(5x/2) - \sin(3x/2)\big) + \cdots \\ &\cdots + \big(\sin((N+1/2)x) - \sin((N-1/2)x)\big)\Big)\,.\end{aligned}$$

Thus,

$$\sin(x/2)D_N(x) = \frac{\sin\left((N+\frac{1}{2})x\right)}{2}\,.$$

Obviously $D_N(0) = N + \frac{1}{2}$. For all other values of x between $-\pi$ and π, we can write

$$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}\,. \tag{21}$$

Since L'Hospital's rule shows that $D_N(x) \to N + \frac{1}{2}$ as $x \to 0$, we shall simply use Eq. (21) as the definition of $D_N(x)$, ignoring the fact that it becomes indeterminate at multiples of 2π.

Exercise 9. Prove that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1\,.$$

(Just use the expression for $D_N(t)$ as a sum of cosines. This would be hard to prove from the more compact expression we just derived for it.)

We now have our desired expression for the partial sum of the Fourier series of an integrable function $f(x)$. If

$$S_N(f;x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx)\,,$$

then

$$S_N(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\,dt\,. \tag{22}$$

Remark. The operation shown in Eq. (22) is called the *convolution* of f and D_N, and usually written $f * D_N(x)$. We will not make any use of this notation.

We now wish to study the difference $f(x) - S_N(f;x)$ and find conditions that will guarantee that it tends to zero. To do so, we shall invoke Exercise 9 above and write this expression as

$$f(x) - S_N(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x) - f(t)\right]D_N(x-t)\,dt\,. \tag{23}$$

Essentially all of Fourier analysis depends on one simple fact, or rather, the technique used to prove that fact:

Riemann Localization Principle. *If $f(x)$ and $g(x)$ are two integrable functions that are equal on some interval $[c-\delta, c+\delta]$, then the difference of their Fourier series tends to zero at c, that is,*

$$\lim_{N\to\infty} S_N(f;c) - S_N(g;c) = 0\,.$$

Proof: Since $S_N(f;c) - S_N(g;c) = S_N(f-g;c)$, we simply consider the function $h = f - g$, which by assumption is an integrable function that is equal to zero on $[c-\delta, c+\delta]$. We wish to show that $S_N(h;c)$ tends to zero. There is no loss in generality in taking $c = 0$, since doing so amounts to considering the function $h_c(x) = h(x-c)$ instead of h. In that case the function

$$r(x) = \begin{cases} h(x)/(2\sin(x/2))\,, & \text{if } \delta < |x| \le \pi\,, \\ 0\,, & \text{if } |x| \le \delta\,, \end{cases}$$

is an integrable function, and we have

$$S_N(h;0) = \frac{1}{\pi}\int_{-\pi}^{\pi} r(t)\sin\Big(N+\frac{1}{2}\Big)t\,dt\,.$$

But this last expression tends to zero by the Riemann–Lebesgue lemma. The proof is now complete.

The reason for calling this result a localization principle is that is says the convergence or divergence of the Fourier series of $f(x)$ at the point c depends only on the values of $f(x)$ on any interval $[c-\delta, c+\delta]$, no matter how short. The values may be changed elsewhere in a more or less arbitrary manner (as long as $f(x)$ remains integrable over $[-\pi,\pi]$), resulting in a new Fourier series, but the convergence or divergence of the new series at c will be the same as for the original function.

2. A convergence theorem. The technique of breaking the interval $[-\pi,\pi]$ into two parts, namely an interval $[c-\delta, c+\delta]$ and its complement is frequently used to establish the convergence of a Fourier series. The integral over the complement always tends to zero, just as in the case above, since $\frac{1}{2\sin((c-t)/2)}$ is bounded outside the interval $[c-\delta, c+\delta]$. If the function

$$k(c;t) = \frac{f(c)-f(t)}{2\sin\big((c-t)/2\big)} \tag{24}$$

is an integrable function of t, it follows from the Riemann–Lebesgue lemma that $S_N(f;c)$ tends to $f(c)$, since

$$f(c) - S_N(f;c) = \frac{1}{\pi}\int_{-\pi}^{\pi} k(c;t)\sin\Big(N+\frac{1}{2}\Big)(c-t)\,dt\,.$$

In particular, this is the case if $f'(c)$ exists, since the function $k(c,t)$ remains bounded as $t \to c$. (Use L'Hospital's rule to prove that this function has the limit $f'(c)/2$ as $t \to c$.) Thus we have the only convergence theorem we are going to prove:

Theorem. *Let $f(t)$ be an integrable function of period 2π such that the derivative $f'(c)$ exists at some point c. Then the Fourier series of $f(t)$ converges to $f(c)$ at $t = c$.*

Much more sophisticated theorems are known, but this one will do as an introduction. In general, because of the localization principle, convergence theorems are nearly always proved by considering the integral of $\big(f(x) - f(t)\big)D_N(x-t)$ over some interval of t near x. The integral over the rest of the period automatically tends to zero.

Exercise 10. Consider the function

$$f(x) = \begin{cases} -1\,, & \text{if } -\pi < x < 0\,, \\ 0\,, & \text{if } x = 0 \text{ or } x = \pm\pi\,, \\ +1\,, & \text{if } 0 < x < \pi\,. \end{cases}$$

Compute the Fourier coefficients of this function. (*Hint*: $a_n = 0$ for all n, so it's not difficult to do.) By our general theorem, the Fourier series converges to $f(x)$ except at multiples of π. What happens at those points? (The graphs of $f(x)$ and $S_{10}(f;x)$ are shown in Fig. 9.)

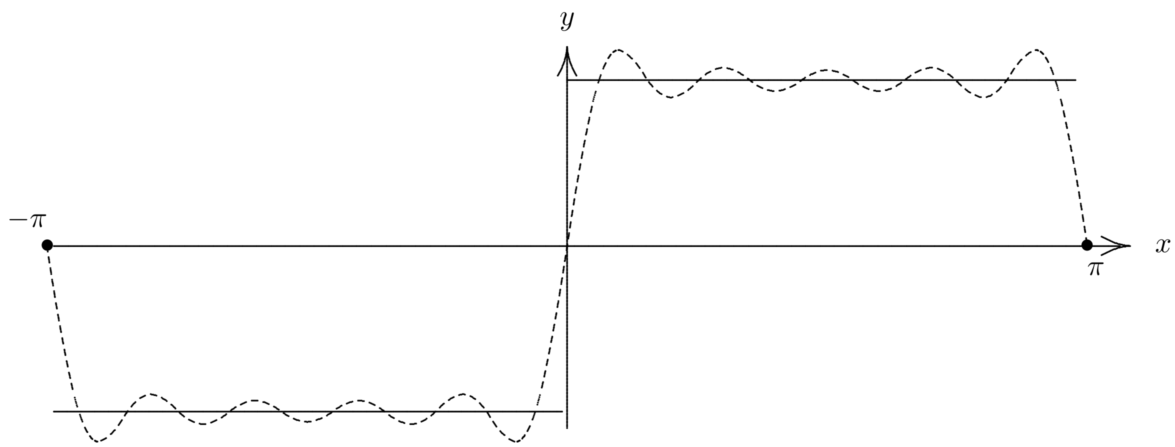

Fig. 9. Graph of a discontinuous function and a partial sum of its Fourier series