Math 333 Homework Assignment 1
J. Sands Due 9/15/09

Remember that collaboration is encouraged, but you must write up solutions in your own words. Style and clarity of exposition are important elements to be considered in your solutions.

0. (Done in class) Show that the collection of subsets of $\mathbb{R}$ defined by
$\mathcal{S}_1 = \{A \subset \mathbb{R} : A \text{ is expressible as a countable union of intervals}\}$ is not a σ-algebra of subsets of $\mathbb{R}$.

1. Show that the classroom definition of an algebra of subsets of X is equivalent to the textbook definition.

2. If $\{\mathcal{S}_\nu : \nu \in V\}$ is a family of σ-algebras of subsets of X, show that $\cap_{\nu \in V} \mathcal{S}_\nu$ is itself a σ-algebra. (Thus the definition of the σ-algebra generated by a collection of subsets $\mathcal{F}$ of X on page 97 makes sense.)

3. page 97, number 3.

4. page 27, number 11

5. Show that the σ-algebra $\mathcal{S}$ generated by the collection S of all intervals in $\mathbb{R}$ is the same as the σ-algebra $\mathcal{B}$ of all Borel sets of $\mathbb{R}$. You may use the Lindelöf Theorem (7.1) which implies that every open set in $\mathbb{R}$ may be expressed as a countable union of open intervals.

6. This problem shows how the definition of an algebra in measure theory fits in with the algebraists' definition of an $\mathbb{F}_2$-algebra.

An $\mathbb{F}_2$-algebra is a ring R which is also a vector space over $\mathbb{F}_2$ such that $c(rs) = (cr)s = r(cs)$ for each $c \in \mathbb{F}_2$ and $r, s \in R$.

Given two subsets A and B of a set X, the symmetric difference of A and B is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

a) Suppose that $\mathcal{S}$ is an algebra of subsets of a set X and we define two binary operations on S:

$A + B = A \triangle B$ (Addition is defined by symmetric difference.)

$A \cdot B = A \cap B$ (Multiplication is defined by intersection.)

Show that $\mathcal{S}$ is a commutative ring with identity element X and zero element $\emptyset$.

b) Define the action of $\mathbb{F}_2$ on $\mathcal{S}$ by

$0 \cdot A = \emptyset$, $1 \cdot A = A$.

Show that $\mathcal{S}$ is an $\mathbb{F}_2$-algebra.

c) Show conversely that if $\mathcal{S}$ is a collection of subsets of X which is an $\mathbb{F}_2$-algebra under the operations of $\triangle$ and $\cap$, and $X \in \mathcal{S}$, then $\mathcal{S}$ is an algebra of subsets of X.