Math 52 Homework Assignment 4 Solutions
J. Sands First Draft Due 10/5/09

1. Write down all subsets of $\{a, b, c, d\}$
Solution:

$$\{\}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}$$

2. Let $S = \{x \in \mathbb{R} : x > 7\}$ and $T = \{x \in \mathbb{R} : x^3 > 100\}$. Prove that $S \subset T$.

Solution: Suppose that s is in S. Then $s > 7$, by the definition of S. So $s^3 > 7^3 = 343 > 100$, by raising both sides of the inequality to the third power and using rules for inequalities which preserve the direction of the inequality for positive numbers. Now we see that $s^3 > 100$ by the transitive property of inequalities. Finally, we conclude that $s \in T$ by the definition of T. We have shown that if s is in S, then s is in T. This is exactly the definition of what it means to say that $S \subset T$, so this is proved.

3. Now let $S = \{x \in \mathbb{R} : x > 7\}$ and $W = \{x \in \mathbb{R} : x^3 > 343\}$. Prove that $S = W$.

Solution: By definition of S, the real number s is in S if and only if $s > 7$. Raising both sides of the inequality to the third power and using the fact that both sides are positive, we see that $s^3 > 343$. Conversely, if $s^3 > 343$, we can take the cube root of both sides and deduce that $s > 343$. So $s > 7$ if and only if $s^3 > 343$. Finally, $s^3 > 343$ if and only if s is in W, by the definition of W. Combining these steps shows that s is in S if and only if s is in W. This is exactly the definition of what it means to say that $S = W$, so this is proved.

4. Suppose A and B are subsets of a set U (sometimes called the universal set). Let A^c denote the complement of A in U and similarly let B^c denote the complement of B in U. The definition of A^c is $A^c = \{x \in U : x \notin A\}$. Show that the complement of $A \cup B$ is $A^c \cap B^c$, that is: $(A \cup B)^c = A^c \cap B^c$. This is also referred to as one of DeMorgan's laws, so that may give you some idea of what you need to use in order to prove it.

Solution: By definition of complement, x is in $(A \cup B)^c$ if and only if $\neg(x \in A \cup B)$. By definition of union, this is equivalent to $\neg(x \in A \vee x \in B)$. By DeMorgan's laws of logic, this is equivalent to $(\neg(x \in A)) \wedge (\neg(x \in B))$. By definition of complement two more times, this is equivalent to $x \in A^c \wedge x \in B^c$. By definition of intersection, this is equivalent to $x \in (A^c \cap B^c)$. Combining the steps above, we see that x is in $(A \cup B)^c$ if and only if $x \in (A^c \cap B^c)$. This is exactly the definition of $(A \cup B)^c = A^c \cap B^c$, so this is proved.

5. Suppose that $S = \{x \in X : P(x)\}$ and $T = \{x \in X : Q(x)\}$. Show that $S \cap T = \{x \in X : P(x) \wedge Q(x)\}$.

Solution: Let $W = \{x \in X : P(x) \wedge Q(x)\}$. By definition of intersection, s is in $S \cap T$ if and only if $s \in S \wedge s \in T$. By definition of S and T, this is equivalent to $P(s) \wedge Q(s)$. By definition of W, this is equivalent to $s \in W$. Combining the steps above, we see that s is in $S \cap T$ if and only if $s \in W$. This is exactly the definition of $S \cap T = W$, so this is proved.

6. Suppose that A, B, and C are subsets of U Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by showing that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$, and using the definitions of union and intersection.

Solution: By definition of intersection $x \in A \cap (B \cup C)$ if and only if $x \in A \wedge x \in (B \cup C)$. By definition of union, this is equivalent to $x \in A \wedge (x \in B \vee x \in C)$. By the distributive laws of logic, this is equivalent to $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$. By the definition of intersection, used twice, this is equivalent to $(x \in A \cap B) \vee (x \in A \cap C)$. By the definition of union, this is equivalent to $x \in (A \cap B) \cup (A \cap C)$. Combining the steps above, we see that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ because we have verified the definition of what it means for two sets to be equal.

7. Let $A = \{n \in \mathbb{Z}^+ : n|42\}$ and $B = \{n \in \mathbb{Z}^+ : n|70\}$.
a. List the elements of A.
b. List the elements of B.
c. List the elements of $A \cup B$.
d. List the elements of $A \cap B$.
e. Show that $|A| + |B| = |A \cap B| + |A \cup B|$.
f. (Extra credit) Show that the equation in part (e) is true for any two finite sets A and B.

Solution:

a. $A = \{1, 2, 3, 6, 7, 14, 21, 42\}$.
b. $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$.
c. $A \cup B = \{1, 2, 3, 5, 6, 7, 10, 14, 21, 35, 42, 70\}$.
d. $A \cap B = \{1, 2, 7, 14\}$.
e. We find $|A|$ by counting the number of elements in A, which is 8. Similarly $|B| = 8$, so $|A| + |B| = 8 + 8 = 16$. From the answers above, we see that $|A \cup B| = 12$ and $|A \cap B| = 4$. Adding gives $|A \cap B| + |A \cup B| = 12 + 4 = 16$. This shows that both sides of the equation equal 16, so the equation is true.
f. If we list the elements of A and then the elements of B, the length of our list will be $|A| + |B|$, by the definition of $|A|$ and $|B|$. The list will consist of the elements of $A \cup B$, along with the elements of $A \cap B$ listed a second time. The reason is that the elements of $A \cap B$ are in both A and B, by definition of intersection. Thus they are listed once as an element of A, and again as an element of B. This gives us another way of counting the length of the list, which yields $|A \cup B| + |A \cap B|$. Since the two ways of determining the length of the list must give the same answer, we conclude that $|A| + |B| = |A \cap B| + |A \cup B|$.

8. Let $S = \{n^2 : n \in \mathbb{Z}^+\}$ and $T = \{m \in \mathbb{Z}^+ : \sqrt{m} \in \mathbb{Z}\}$ Show that $S = T$.

Solution: If $s \in S$, then $s = n^2$ and $n \in \mathbb{Z}^+$, by the definition of S. By taking (positive) square roots, this implies that $\sqrt{s} = n \in \mathbb{Z}$. We find that $s \in T$, by the definition of T. Since $s \in S$ implies that $s \in T$, we now know that $S \subset T$, by definition of subset. Also, if $t \in T$, then $\sqrt{t} \in \mathbb{Z}$. Since we are taking the positive square root, this shows that $\sqrt{t} = n \in \mathbb{Z}^+$. Squaring both sides, we get $t = n^2$ for some $n \in \mathbb{Z}^+$. Now we see that $t \in S$, by the definition of S. Since we have shown that $t \in T$ implies $t \in S$, we know that $T \subset S$, by definition of subset. Finally, since $S \subset T$ and $T \subset S$, we conclude that $S = T$, by the definition of equality of sets.

9a. Write in set notation"W is the set of all real solutions to the equation $x^4 - 4x^2 + 3 = 0$".
b. Show that $W = \{1, -1, \sqrt{3}, -\sqrt{3}\}$

Solution:

a. $W = \{x \in \mathbb{R} : x^4 - 4x^2 + 3 = 0\}$
b. Substituting $x = 1$ into $x^4 - 4x^2 + 3$ results in a value of $1^4 - 4 \cdot 1^2 + 3 = 0$. Thus $1 \in W$, by definition of W. Similarly, we can check that -1, $\sqrt{3}$, and $-\sqrt{3}$ are in W. A polynomial of degree 4 can have at most 4 roots, so there are no other solutions to the equation defining W. Thus $W = \{1, -1, \sqrt{3}, -\sqrt{3}\}$