NAME:

Math 52C Solutions for Practice Test I October 14, 2009

INTEGERS

1a. Use the Euclidean algorithm to find GCD(255,68).

$$255 = 68(3) + 51$$
$$68 = 51(1) + 17$$
$$51 = 17(3) + 0$$

GCD(255,68)=17.

b. Are 255 and 68 relatively prime?

To say that they are relatively prime means that their GCD equals 1. These two numbers have a GCD of 17, not 1, so they are not relatively prime.

GEOMETRIC SERIES: PROVING AND DISPROVING

2a. Write the negation of the following statement in a way that changes the quantifier. "There exists a positive integer n such that $6^n - 1$ is not a multiple of 5.

For all positive integers n, $6^n - 1$ is a multiple of 5.

b. Prove the statement that you have just written in part a.

The formula for the sum of a geometric progression says that

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

Setting $r = 6$, we get

$$1 + 6 + 6^2 + \cdots + 6^{n-1} = \frac{1 - 6^n}{1 - 6} = \frac{6^n - 1}{5}.$$

Multiplying through by 5 on both sides yields

$$5(1 + 6 + 6^2 + \cdots + 6^{n-1}) = 6^n - 1.$$

Since $k = 1 + 6 + 6^2 + \cdots + 6^{n-1}$ is an integer, we see that $6^n - 1 = 5k$, so $6^n - 1$ is a multiple of 5.

c. Write the negation of the following statement in a way that changes the quantifier. "For all positive integers n, the integer $6^n - 2$ is a multiple of 4.

There exists a positive integer n such that $6^n - 2$ is not a multiple of 4.

d. Prove the statement that you have just written in part c.

Choosing $n = 2$, we see that $6^2 - 2 = 36 - 2 = 34$, which is not a multiple of 4 because $34 = 4(8) + 2$.

LOGIC

3a. Prove that $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$. What is this rule called?

Two logical statements are equivalent if they have the same truth table. We construct a truth table using the rules for $\wedge$ and $\vee$.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

The columns for $P \wedge (Q \vee R)$ and for $(P \wedge Q) \vee (P \wedge R)$ are the same. This shows that these two statements are equivalent. This is one of the distributive laws for logical statements.

3b. Show that $P \wedge (Q \vee P)$ is equivalent to P.

We construct a truth table.

P	Q	$Q \vee P$	$P \wedge (Q \vee P)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**F**
F	F	F	**F**

The columns for P and for $P \wedge (Q \vee P)$ are the same. This shows that the two statements are equivalent.

4a. Write down two different ways of expressing the negation of the following statement about a real number b. "Either $b > 2$ or $b < -2$."

DeMorgan's law says that the negation of $(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$. We apply this here. "It is not the case that either $b > 2$ or $b < -2$."

Both $b \leq 2$ and $b \geq -2$.

b. Write down the converse of the following statement. "If the polynomial $x^2 + bx + 1$ has two real roots then $b > 2$ or $b < -2$."

The converse of $P \implies Q$ is $Q \implies P$. We apply this here. "If $b > 2$ or $b < -2$ then $x^2 + bx + 1$ has two real roots."

c. Write down the contrapositive of the following (same) statement. "If the polynomial x^2+bx+1 has two real roots then $b > 2$ or $b < -2$."

The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$. The two are equivalent. We apply this here. " If $b \leq 2$ and $b \geq -2$, then $x^2 + bx + 1$ does not have two real roots."

d. Prove that the polynomial $x^2 + bx + 1$ has two real roots if and only if $b > 2$ or $b < -2$. Notice that this can be done by proving the statement you have written down in part (b) and the statement you have written down in part (c). You may use the quadratic formula.

By the quadratic formula, the roots of $x^2 + bx + 1$ are $(-b \pm \sqrt{b^2 - 4})/2$.

If $b > 2$ or $b < -2$, then by multiplying inequalities, we get $b^2 > 4$. Subtracting 4 from both sides shows that $b^2 - 4 > 0$, so $\sqrt{b^2 - 4}$ is a positive real number. Now we can see that the quadratic formula gives us two distinct real roots $(-b \pm \sqrt{b^2 - 4})/2$. This proves the statement written in 4b.

If $b \leq 2$ and $b \geq -2$, then by multiplying inequalities, we get $b^2 \leq 4$. Subtracting 4 from both sides shows that $b^2 - 4 \leq 0$. This implies that $\sqrt{b^2 - 4}$ is 0 or an imaginary number. In this case, the two roots $(-b \pm \sqrt{b^2 - 4})/2$ are either identical or imaginary. So we do not get two distinct real roots. This proves the statement written in 4c.

SETS

5. Define $S = \{n \in I\!N : 30|n\}$, $T = \{n \in I\!N : 5|n\}$ and $W = \{n \in I\!N : 3|n\}$.

a. Show that $S \subset T$.

The definition of $S \subset T$ is $s \in S \implies s \in T$. So we take $s \in S$ as our hypothesis, and show how to conclude that $s \in T$.

If $s \in S$, then $30|s$, by the definition of S. Then $s = 30k$ for some positive integer k, by the definition of "divides". Using the associative law, we get $s = 5(6k)$. This shows that $5|s$ by the definition of "divides". Since $5|s$, we conclude that $s \in T$, by the definition of T. Assuming $s \in S$, we have concluded that $s \in T$. This is exactly what it means to say that $S \subset T$, so this is proved.

b. Show that $S \subset W$.

If $s \in S$, then $30|s$, by the definition of S. Then $s = 30k$ for some positive integer k, by the definition of "divides". Using the associative law, we get $s = 3(10k)$. This shows that $3|s$ by the definition of "divides". Since $3|s$, we conclude that $s \in W$, by the definition of W. Assuming $s \in S$, we have concluded that $s \in W$. This is exactly what it means to say that $S \subset W$, so this is proved.

c. Show that $S \subset (T \cap W)$.

The definition of $T \cap W$ is $x \in T \cap W \iff (x \in T \wedge x \in W)$ If $s \in S$, we have proved above that $s \in T$ and $s \in W$. By the definition of intersection, this means that $s \in T \cap W$. So $s \in S$ implies $s \in T \cap W$. This is exactly the definition of the statement $S \subset T \cap W$, so it is proved.

d. Show that $S \neq (T \cap W)$.

Two sets are equal if and only if each one contains the other. We show that this is not true in this case because S does not contain $T \cap W$. Note that $15 \notin S$, but $5 \cdot 3 = 15 \in T$ and $3 \cdot 5 = 15 \in W$. Thus $15 \in T \cap W$, by definition of intersection. But $15 \notin S$ since $30 \not| 15$. Since $15 \in T \cap W$ while $15 \notin S$, we have proved that $S \neq (T \cap W)$.

e. Show that S is closed under addition. What rules of arithmetic are used?

To say that S is closed under addition means that if $s_1 \in S$, and $s_2 \in S$, then $s_1 + s_2 \in S$. So we assume $s_1 \in S$, and $s_2 \in S$, we get from the definition of S that $s_1 = 30k_1$ and $s_2 = 30k_2$, for integers k_1 and k_2. Adding and using the distributive law, we get $s_1 + s_2 = 30k_1 + 30k_2 = 30(k_1 + k_2) = 30k_3$. We have put $k_3 = k_1 + k_2$, which we know is an integer because it is the sum of two integers. Hence $s_1 + s_2 = 30k_3$, and this shows that $30|s_1 + s_2$. Since S consists of all multiples of 30 by definition, we conclude that $s_1 + s + 2 \in S$. From the hypothesis that $s_1 \in S$ and $s_2 \in S$, we have reached the conclusion that $s_1 + s_2 \in S$. This shows that S is closed under addition.

f. Show that S is not closed under division.

Notice that 60 and 30 are both elements of S, but $60/30 = 2$ is not an element of S because it is not a multiple of 30. We have divided two elements of S and obtained an element which is not in S. This is enough to show that S is not closed under division.

RELATIONS

6. Suppose $S = I\!N$, and $\rho = \{(s,t) \in S \times S : s \pmod 5 = t \pmod 5\}$.

   a. Show that ρ is an equivalence relation.

We must show that ρ is reflexive, symmetric, and transitive.

Reflexive means that for each $s \in S$, we must have $(s,s) \in \rho$. So suppose that $s \in S$. Then $s \pmod 5 = s \pmod 5$, so $(s,s) \in \rho$.

Symmetric means that if $(s,t) \in \rho$ then $(t,s) \in \rho$. So suppose that $(s,t) \in \rho$. Then using the definition of this particular relation ρ, we have $s \pmod 5 = t \pmod 5$. This says that two integers are equal, so if we change the order, they are still equal, and $t \pmod 5 = s \pmod 5$. Using the definition of the relation again, we see that $(t,s) \in \rho$.

Transitive means that if $(s,t) \in \rho$ and $(t,w) \in \rho$, then $(s,w) \in \rho$. So suppose that $(s,t) \in \rho$ and $(t,w) \in \rho$. Using the definition of this particular relation ρ, we have $s \pmod 5 = t \pmod 5$ and $t \pmod 5 = w \pmod 5$. Combining these two equalities of integers, we can see that $s \pmod 5 = w \pmod 5$. The last equation says that $(s,w) \in \rho$.

   b. Write down the equivalence class of 17 for this relation.

$[17] = \{2, 7, 12, 17, 22, 27, \cdots\}$

   c. Write down the equivalence class of 15 for this relation.

$[15] = \{5, 10, 15, 20, 25, \cdots\}$

   d. We define a binary relation on $I\!N$ by $\sigma = \{(a,b) : a|b\}$. Show that σ is reflexive and transitive, but not symmetric.

If $a \in I\!N$, then $a|a$ since $a = 1 \cdot a$. This shows that $(a,a) \in \sigma$, so σ is reflexive.

If $(a,b) \in \sigma$ and $(b,c) \in \sigma$, then the definition of sigma tells us that $a|b$ and $b|c$. Writing out what this means, we get $b = ak$ and $c = bm$, for some integers k and m. Substituting the first equation in the second, we get $c = akm$. This shows that $a|c$. According to the definition of σ, we can then say that $(a,c) \in \sigma$. Since the assumption that $(a,b) \in \sigma$ and $(b,c) \in \sigma$ together imply that $(a,c) \in \sigma$, we know that σ is transitive.

A single example is enough to show that σ is not symmetric. A simple one is based on the fact that $2|4$ but $4 \not| 2$. Thus $(2,4) \in \sigma$, but $(4,2) \notin \sigma$. This shows that σ is not symmetric.