Math 52C Solutions to Practice for Test II on Wednesday, November 11, 2009

1. FUNCTIONS

a. According to our formal definition, what is a function?
A function is a relation f from X to Y with the property that for each $x \in X$ there is one and only one $y \in Y$ such that $(x, y) \in f$.

b. Let $S = \{2, 4, 6\}$ and $T = \{7, 8, 9, 10\}$. Define a relation σ from S to T by $\sigma = \{(2, 8), (4, 7), (6, 9)\}$. Show that σ is a function.
For each of the three elements in S, there is one and only one element of T paired with it: $2 \in S$ is paired with $8 \in T$, and no other element, $4 \in S$ is paired with $7 \in T$, and no other element, while $6 \in S$ is paired with $9 \in T$ and no other element. This accounts for each of the elements of S.

c. What is the definition of one-to-one? A function f is one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

d. Show that σ is one-to-one.
Each of the three inputs 2, 4, and 6, produces a different output, namely 8, 7, and 9, respectively. So we do not have two different inputs producing the same output. Thus σ satisfies the definition of one-to-one.

e. Why doesn't σ have an inverse function from T to S?
A function has an inverse if and only if it is one-to-one and onto. This function σ is not onto, so it has no inverse. There is no input that produces $10 \in T$ as an output.

f. Now suppose that we consider σ to be a function from $S = \{2, 4, 6\}$ to $R = \{7, 8, 9\}$. Write down the inverse function of σ from R to S.
$\sigma^{-1} = \{(8, 2), (7, 4), (9, 6)\}$.

g. Suppose that f and g are functions for which we can define the composition $f \circ g$. If f and g are one-to-one, prove that $f \circ g$ is one-to-one.
If $f \circ g(z_1) = f \circ g(z_2)$, then by definition of composition, $f(g(z_1) = f(g(z_2)$. Since f is one-to-one, we can conclude that $g(z_1) = g(z_2)$. Now since g is one-to-one, we can conclude that $z_1 = z_2$. We have shown that $f \circ g(z_1) = f \circ g(z_2)$ implies $z_1 = z_2$. Thus $f \circ g$ satisfies the definition of one-to-one.

h. If f and g have inverse functions, prove that $f \circ g$ has an inverse, namely $g^{-1} \circ f^{-1}$.
Suppose z_1 is an element in the domain of g. Let $g(z_1) = x_1$ and $f(x_1) = y_1$. Based on this, we have $f \circ g(z_1) = y_1$. Also, $g^{-1}(x_1) = z_1$ and $f^{-1}(y_1) = x_1$, by the definitions of composition and inverse. Then $g^{-1} \circ f^{-1}(y_1) = g^{-1}(f^{-1}(y_1) = g^{-1}(x_1) = z_1$. Since z_1 can be any element in the domain of g, we see that $g^{-1} \circ f^{-1}(y_1) = z_1$ whenever $f \circ g(z_1) = y_1$. Similarly, if $g^{-1} \circ f^{-1}(y_1) = z_1$, we can show that $f \circ g(z_1) = y_1$. Thus $g^{-1} \circ f^{-1}$ satisfies the definition of the inverse function of $f \circ g$.

Suppose that $X = \mathbb{R}^+$ is the set of positive real numbers and $Y = \mathbb{N}$ is the set of positive integers, and f is the function from $\mathbb{R}^+$ to $\mathbb{N}$ that rounds a real number up to the closest integer greater than or equal to it. So $f(42.1) = 43$, $f(7.9) = 8$, $f(10) = 10$, etc.

i. Is f one-to-one? Explain.
No. For example $f(7.9) = 8$, and $f(7.7) = 8$. Two different inputs, 7.9 and 7.7, produce the same output of 8, so the function is not one-to-one.

j. Show that f is onto.
Any element in the set Y is a natural number n. Notice that $f(n) = n$, so n is an output of the function. This shows that every element in Y is an output of the function, which is what it means for the function f to be onto.

k. Does f have an inverse function? Explain.
A function has an inverse if and only if it is one-to-one and onto. The function f does not have an inverse function because it is not onto.

2. COMPLETE INDUCTION AND RECURSION

a. A sequence is defined by setting $a_1 = 4$, $a_2 = 10$, and $a_{n+1} = 4a_n - 3a_{n-1}$ for each integer $n \geq 2$. Show by complete induction that $a_n = 3^n + 1$ for all positive integers n.
The proof is by complete induction on n. Let $P(n)$ be the statement that $a_n = 3^n + 1$.
Basis Step. $P(1)$ says that $a_1 = 3^1 + 1$. This is true because both sides of the equation are equal to 4.
$P(2)$ says that $a_2 = 3^2 + 1$. This is true since both sides of the equation are equal to 10.
Inductive Step. We use $P(N-1)$ and $P(N)$ to prove $P(N+1)$. First, $a_{N+1} = 4a_N - 3a_{N-1}$ by definition of a_N. By the inductive hypothesis, we may use $a_N = 3^N + 1$ and $a_{N-1} = 3^{N-1} + 1$. Substituting these in and simplifying algebraically, we get $a_{N+1} = 4(3^N + 1) - 3(3^{N-1} + 1) = 4(3^N) + 4 - 3(3^{N-1}) - 3 = 4(3^N) - 3^N + 1 = (4-1)3^N + 1 = 3(3^N) + 1 = 3^{N+1} + 1$. Combining all of these equations, we see that $a_{N+1} = 3^{N+1} + 1$, which establishes $P(N+1)$ and completes the proof by induction.

b. Find a formula for b_n if $b_1 = 6$, $b_2 = 18$, and $b_{n+1} = 4b_n - 3b_{n-1}$ for each integer $n \geq 2$.
As in the homework problem 6 of section 2.2, we find σ_1 and σ_2 as the roots of the quadratic equation $x^2 = 4x - 3$. We solve this by $0 = x^2 - 4x + 3 = (x-3)(x-1)$, so $\sigma_1 = 3$ and $\sigma_2 = 1$. Our solutions will be of the form $b_n = A \cdot 3^n + B \cdot 1^n$. We find A and B by setting $6 = a_1 = A \cdot 3 + B \cdot 1$, and $18 = a_2 = A \cdot 3^2 + B \cdot 1^2$. This leads to $A = 2$, and $B = 0$, so $b_n = 2 \cdot 3^n$

3. EXTENDED EUCLIDEAN ALGORITHM

a. Use the Euclidean Algorithm to find integer values of x and y such that $86x + 38y = 2$. Show your work.
$86 - 38(2) = 10$
$38 - 10(3) = 8$
$10 - 8(1) = 2$
$8 - 2(4) = 0$. So $GCD(86, 38) = 2$. Now we do back-substitution.
$2 = 10 - 8 = 10 - [38 - 10(3)] = 10(4) - 38 = [86 - 38(2)](4) - 38 = 86(4) - 38(9)$. So we may use $x = 4$ and $y = -9$.

b. What is the GCD of 86 and 38?
We found that it is 2.

c. Find values of z and w so that $86z + 38w = 6$.
Multiplying by 3 on both sides of $86(4) - 38(9) = 2$ gives $86(12) - 38(27) = 6$. So we can use $z = 12$ and $w = 27$.

d. Can you find values of s and t so that $86s + 38t = 7$? Why or why not?
We cannot because the $GCD(86, 38) = 2$ and 7 is not a multiple of 2. There is a theorem that says we can find integers s and t such that $as + bt = d$ if and only if d is a multiple of $GCD(a, b)$.

e. State the Fundamental Theorem of Arithmetic
Every positive integer can be factored into a product of prime numbers in a unique way.

f. Suppose that $b \mid c$ and $a \mid c$ and GCD(a,b)=1. Show that $ab \mid c$.
From the Euclidean algorithm, we know that since $GCD(a, b) = 1$ we can find integers x and y such that $ax + by = 1$. Now multiply both sides by c to get $acx + bcy = c$. Since $b \mid c$, we have $c = bk$ for some integer k. Since $a \mid c$, we have $c = am$ for some integer m. Substituting these into $acx + bcy = c$ gives $abk + bam = c$. So by the commutative and distributive laws, $ab(k + m) = c$. This show that $ab \mid c$.
(It is also possible to prove this by using the Fundamental Theorem of Arithmetic.)