COMPLETE INDUCTION AND RECURRENCE RELATIONS
Math 52C Fall 2009
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You want to prove that a certain statement is true for all positive integers **n** greater than or equal to 1. Identify what that statement is before writing the proof. We will call it P(n). The proof consists of showing first that P(1) and P(2) are true and then that P(N) and P(N-1) together imply P(N+1) for each N>1.

Problem: A sequence is defined by $a_1=1$, $a_2=7$, and the recurrence relation $a_{n+1}=7a_n-12a_{n-1}$ for each integer n>1. Prove that $a_n=4^n-3^n$ for each integer n>0.

Solution: The proof is by induction on n.

P(n) will be the statement that $a_n=4^n-3^n$.

1) **Basis Step: We prove P(1) and P(2).**

   P(1) says $a_1=4^1-3^{1.}$ The left hand side is $a_1=1$, because this was given. The right hand side is 4-3=1. These two answers are equal, so P(1) is true.

   P(2) says $a_2=4^2-3^2$. The left hand side is $a_2=7$, because this was given. The right hand side is $4^2-3^2=16-9=7$. These two answers are equal, so P(2) is true.

2) **Inductive Step: For the inductive hypothesis, we assume that both P(N) and P(N-1) are true. These say that** $a_N=4^N-3^N$ and $a_{N-1}=4^{N-1}-3^{N-1}$ **. We will use these to prove P(N+1), which says** $a_{N+1}=4^{N+1}-3^{N+1}$ **.**

   Starting with the left hand side of the equation in P(N+1), we will show by a sequence of steps that it is equal to the right hand side.
   By the way the sequence was defined, $a_{N+1}=7a_N-12a_{N-1.}$
   By the inductive hypothesis, we may substitute in $a_N=4^N-3^N$ and $a_{N-1}=4^{N-1}-3^{N-1}$**.**
   We get $a_{N+1}=7(4^N-3^N)-12(4^{N-1}-3^{N-1})$ which simplifies algebraically to
   $7[4(4^{N-1})-3(3^{N-1})]-12(4^{N-1}-3^{N-1}) = 28(4^{N-1})-21(3^{N-1})-12(4^{N-1})+12(3^{N-1})$
   $= (28-12)(4^{N-1})+(12-21)(3^{N-1}) = (16)(4^{N-1})+(-9)(3^{N-1})=4^2(4^{N-1})-3^2(3^{N-1}) =$
   $(4^{N-1+2})-(3^{N-1+2}) = 4^{N+1}-3^{N+1}$.
   Combining these equalities shows that $a_{N+1}= 4^{N+1}-3^{N+1}$.

   We have proved P(N+1) under the assumption of P(N) and P(N-1). By the Principal of Mathematical Induction, we conclude that P(n) is true for all integers n≥1.