

***L*-FUNCTIONS AT THE ORIGIN AND
ANNIHILATION OF CLASS GROUPS
IN MULTIQUADRATIC EXTENSIONS**

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ABSTRACT. Fix a Galois extension K/F of number fields such that the Galois group G has exponent 2. Let S be a finite set of primes of F containing the infinite primes and all those which ramify in K , let S_K denote the primes of K lying above those in S , and let \mathcal{O}_K^S denote the ring of S_K -integers of K . We obtain non-trivial annihilators for the class group of \mathcal{O}_K^S from the equivariant L -function $\theta_{K/F}^S(s)$ for K/F and the set S , regardless of the order of vanishing at the origin.

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I. INTRODUCTION

Fix an abelian Galois extension of number fields K/F and let G denote the Galois group. Also fix a finite set S of primes of F which contains all of the infinite primes of F and all of the primes which ramify in K . Since it is fixed throughout, we will often suppress S in the notation. Associated with this data is an equivariant L -function, $\theta_{K/F}(s) = \theta_{K/F}^S(s)$, a meromorphic function of $s \in \mathbb{C}$ with values in the group ring $\mathbb{C}[G]$. When the real part of s is greater than 1 it is defined as a product over the (finite) primes \mathfrak{p} of F that are not in S . Let $N\mathfrak{p}$ denote the absolute norm of the ideal \mathfrak{p} and $\sigma_{\mathfrak{p}} \in G$ denote the Frobenius automorphism of \mathfrak{p} . Then

$$\theta_{K/F}^S(s) = \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{N\mathfrak{p}^s} \sigma_{\mathfrak{p}}^{-1}\right)^{-1}.$$

Each component of this function extends meromorphically to all of \mathbb{C} , and its behavior at $s = 0$ is connected with the arithmetic of K .

The ring of S -integers \mathcal{O}_F^S of F is defined to be the set of elements of F whose valuation is non-negative at every prime not in S . When $K = F$, the function $\theta_{F/F}^S(s)$ is simply the identity automorphism of F times $\zeta_F^S(s)$, the Dedekind zeta-function of F with Euler factors for the primes in S removed. The function $\zeta_F^S(s)$ may be viewed as the zeta-function of the Dedekind domain \mathcal{O}_F^S .

Letting S_K denote the set of primes of K lying above those in S , we define \mathcal{O}_K^S to be the ring of S_K -integers of K . Then Cl_K^S denotes the S_K -class group of K , which may be identified with the group of nonzero fractional ideals of \mathcal{O}_K^S modulo principal fractional ideals. Denote the order of Cl_K^S by h_K^S . Let μ_K denote the group of all roots of unity in K , and w_K denote its order. When the Brumer-Stark conjecture holds, it implies that $w_K \theta_{K/F}^S(0)$ annihilates Cl_K^S as a module over the group-ring $\mathbb{Z}[G]$. However, this conjecture is vacuous when $\theta_{K/F}^S(0) = 0$. On the other hand, one knows that for $K = F$, the leading

term in the Taylor series at $s = 0$ for ζ_F^S is $\zeta_F^{S,*} = -h_F^S R_F^S / w_F$, where R_F^S is the regulator of the S -units of F . One sees that this quantity still provides an annihilator $-h_F^S$ for Cl_F^S , upon removing the factors R_F^S and w_F which relate to the group of S -units and its torsion subgroup. In this paper, we obtain results on the annihilation of Cl_K^S by what may be considered the leading term of $\theta_{K/F}^S(s)$ at $s = 0$. Going further, we exhibit annihilators coming from higher order terms as well. Such results are clearly related to the refined Stark conjectures of Rubin and Popescu, but those do not directly concern annihilators for Cl_K^S . Rather, the connection between leading terms of equivariant L -functions and annihilators of class groups appears in more recent conjectures of Burns [2], and results of Buckingham [1] which had their origins in ideas of Snaith [5].

To state our results, let \hat{G} denote the group of characters of G and recall that the S -imprimitive Artin L -functions for a character $\psi \in \hat{G}$ is defined as

$$L_{K/F}^S(s, \psi) = \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{N\mathfrak{p}^s} \psi(\sigma_{\mathfrak{p}})\right)^{-1},$$

so that using the idempotents $e_{\psi} = \frac{1}{|G|} \sum_{\sigma \in G} \psi(\sigma) \sigma^{-1}$, we have

$$\theta_{K/F}^S(s) = \sum_{\psi \in \hat{G}} L_{K/F}^S(s, \psi^{-1}) e_{\psi}.$$

Defining $L_{K/F}^*(\psi) = L_{K/F}^{S,*}(\psi)$ to be the first nonzero coefficient in the Taylor series for $L_{K/F}^S(s, \psi)$ at $s = 0$, one then puts

$$\theta_{K/F}^* = \theta_{K/F}^{S,*} = \sum_{\psi \in \hat{G}} L_{K/F}^{S,*}(\psi^{-1}) e_{\psi}.$$

Next define a regulator as in Burns [2]. For each prime $w \in S_K$, let $|\cdot|_w$ denote the corresponding normalized absolute value on K . Let $U_K = U_K^S = (\mathcal{O}_K^S)^*$, the multiplicative group of S_K -units in K . Let Y_K^S be the free abelian group on primes in S_K . This has a

natural G -action which makes it a $\mathbb{Z}[G]$ -module. The submodule $X_K = X_K^S$ is the kernel of the augmentation homomorphism $Y_K = Y_K^S \rightarrow \mathbb{Z}$ which sends each element to the sum of its coefficients. Then $\mathbb{R}U_K^S = \mathbb{R} \otimes_{\mathbb{Z}} U_K^S$ is known to be isomorphic to $\mathbb{R}X_K^S = \mathbb{R} \otimes_{\mathbb{Z}} X_K^S$ by the \mathbb{R} -linear extension $\lambda_{K,\mathbb{R}} = \lambda_{K,\mathbb{R}}^S$ of the map $\lambda_K = \lambda_K^S : U_K^S \rightarrow \mathbb{R}X_K^S$ defined by

$$\lambda_K(u) = - \sum_{w \in S_K} \log |u|_w \cdot w.$$

Any $\mathbb{Z}[G]$ -module homomorphism $f : M \rightarrow N$, determines an $\mathbb{R}[G]$ -module homomorphism

$$f_{\mathbb{R}} : \mathbb{R}M \rightarrow \mathbb{R}N$$

by extension of scalars. In particular, suppose that we fix a $\mathbb{Z}[G]$ -module homomorphism $f : U_K^S \rightarrow X_K^S$. Since $\mathbb{R}[G]$ is a semisimple commutative ring and $\mathbb{R}U_K^S$ is finitely generated as a module over this ring, there exists a complementary $\mathbb{R}[G]$ -module P such that $\mathbb{R}U_K^S \oplus P$ is a finitely generated free module. Using the identity map 1_P on P , one then obtains a well-defined regulator of f in $\mathbb{R}[G]$:

$$R(f) = \det_{\mathbb{R}[G]}(\lambda_{K,\mathbb{R}}^{-1} \circ f_{\mathbb{R}}) = \det_{\mathbb{R}[G]}((\lambda_{K,\mathbb{R}}^{-1} \circ f_{\mathbb{R}}) \oplus 1_P).$$

Let $r^S(\psi) = r(\psi)$ denote the dimension of the \mathbb{R} -vector space $e_{\psi}\mathbb{R}U_K^S$.

Our main result is the following theorem, proved in a slightly stronger form as Theorem 4.5 at the end of this paper. Remark 4.6 indicates how it may be strengthened further.

Main Theorem. *Let K be a composite of a finite number of quadratic extensions of a number field F . Let S contain the infinite primes of F and those which ramify in K/F . Suppose that $f : U_K^S \rightarrow X_K^S$ is a $\mathbb{Z}[G]$ -module homomorphism with $\ker(f)$ finite. Let $\alpha \in \mathbb{Z}[G]$ annihilate μ_K , and let ψ be an irreducible character of G . Then $|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_{\psi}$ lies in $\mathbb{Z}[G]$ and annihilates Cl_K^S .*

Remark 1. Burns [2] obtains more general results of this form, considering components of the units and of X_K^S for each character separately. His Conjecture 2.5.1 and evidence for it (which includes the multiquadratic extensions considered here) then involves an additional factor of $|G|^2$ in the resulting annihilator. Macias Castillo [4] obtains stronger results specifically for multiquadratic extensions such as those considered here, but not for all characters. We have chosen to show what can be done working with U_K^S ; Burns and Macias Castillo (and others) formulate their results in terms of certain torsion-free subgroups of U_K^S . In a subsequent paper, we will detail the connections between their work and ours more fully.

Remark 2. The principal Stark conjecture [6] states that $|G|^{r^S(\psi)+1} \alpha R(f) \theta_{K/F}^{S,*} e_\psi$ lies in $\mathbb{Q}[G]$, and is already known in the case of multiquadratic extensions.

II. COMPUTING $\theta_{K/F}^* = \theta_{K/F}^{S,*}$

From now on, we will omit the set of primes S from our notation. So $Y_K = Y_K^S$, $X_K = X_K^S$, $U_K = U_K^S$, $h_F = h_F^S$, $r(\psi) = r^S(\psi)$, $R_F = R_F^S$, $L_{K/F} = L_{K/F}^S$, $\zeta_F = \zeta_F^S$, and $\theta_{K/F} = \theta_{K/F}^S$, etc.

Proposition 2.1. For the principal character ψ_0 of $\text{Gal}(K/F)$, we have

$$\theta_{K/F}^* e_{\psi_0} = \frac{-h_F R_F}{w_F} e_{\psi_0}$$

Proof. Since ψ_0 is the inflation of the trivial character on $\text{Gal}(F/F)$, the functorial properties of Artin L -functions give

$$\theta_{K/F}^* e_{\psi_0} = L_{K/F}^*(\psi_0) e_{\psi_0} = \zeta_F^* e_{\psi_0}.$$

The result then follows from the analytic class number formula.

Now assume that $G = \text{Gal}(K/F)$ has exponent 2, and let ψ be a non-trivial character of G . The image of ψ is a non-trivial cyclic group of exponent 2, hence of order 2. So $\ker(\psi)$ has index 2 in G . Let E_ψ denote the fixed field of $\ker(\psi)$, a relative quadratic extension of F . Let $H_{E_\psi/F}$ denote the cokernel of the natural map from Cl_F to Cl_{E_ψ} that is induced by extension of ideals. Let τ_ψ denote the generator of $\text{Gal}(E_\psi/F)$. We will have occasion to fix a lift of τ_ψ to an element of G , which we also denote by τ_ψ . If M is a $\mathbb{Z}[G]$ -module and $\alpha \in \mathbb{Z}[G]$, we let M^α denote the image of M under multiplication by α , and M_α denote the kernel of multiplication by α .

Proposition 2.2.

$$\theta_{K/F}^* e_\psi = \frac{|C_{E_\psi/F}|}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})} \frac{R_{E_\psi}}{R_F} \frac{w_F}{w_{E_\psi}} e_\psi.$$

Proof. First, ψ is induced from the non-trivial character of $\text{Gal}(E_\psi, F)$, and this character is the difference between the regular representation of $\text{Gal}(E_\psi/F)$ and the trivial character. The functorial properties of Artin L -functions and the analytic class number formula then give

$$\theta_{K/F}^* e_\psi = L_{K/F}^*(\psi) e_\psi = \frac{\zeta_{E_\psi}^*(0)}{\zeta_F^*(0)} e_\psi = \frac{h_{E_\psi}}{h_F} \frac{R_{E_\psi}}{R_F} \frac{w_F}{w_{E_\psi}} e_\psi.$$

A computation of Tate ([6, Thm. IV.5.4]) then shows that

$$\frac{h_{E_\psi}}{h_F} = \frac{|C_{E_\psi/F}|}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})},$$

and this completes the proof.

III. COMPUTING $R(f)$

Lemma 3.1. *Suppose that ϕ is an endomorphism of a finitely generated projective R -module M .*

- a. If R' is an overring of R , let $M' = R' \otimes_R M$ and $\phi' = 1_{R'} \otimes_R \phi$, an endomorphism of M' . Then $\det_{R'}(\phi') = \det_R(\phi)$.
- b. If $R = R_1 \oplus R_2$, then consequently $M = M_1 \oplus M_2$ where M_1 is a finitely generated projective R_1 -module and M_2 is a finitely generated projective R_2 -module, and $\phi = \phi_1 \oplus \phi_2$ for ϕ_1 an endomorphism of M_1 and ϕ_2 an endomorphism of M_2 . Then $\det_R(\phi) = (\det_{R_1}(\phi_1), \det_{R_2}(\phi_2)) \in R_1 \oplus R_2 = R$. Using $1 = e_1 + e_2$ where e_1 and e_2 are idempotents of R lying in R_1 and R_2 respectively, this may be written as $\det_R(\phi) = \det_{R_1}(\phi_1)e_1 + \det_{R_2}(\phi_2)e_2$.

Proof.

- a. Choose P so that $M \oplus P$ is a finitely generated free R -module with basis $\{b_1, \dots, b_k\}$, and let $P' = R' \otimes_R P$. Then $M' \oplus P' \cong R' \otimes_R (M \oplus P)$ is a finitely generated free R' -module with basis $\{b'_1 = 1 \otimes b_1, \dots, b'_k = 1 \otimes b_k\}$. Using these bases, it is clear that the matrix of $\phi \oplus 1_P$ is the same as the matrix of $\phi' \oplus 1_{P'}$, as the latter may be identified with $1_{R'} \otimes (\phi \oplus 1_P)$. Thus $\det_{R'}(\phi') = \det_{R'}(\phi' \oplus 1_{P'}) = \det_R(\phi \oplus 1_P) = \det_R(\phi)$.
- b. Note that $M_1 = e_1 M$ and $M_2 = e_2 M$. After choosing P so that $M \oplus P$ is a finitely generated free R -module, we see that $e_1(M \oplus P) = e_1 M \oplus e_1 P = M_1 \oplus e_1 P$ is a finitely generated free R_1 -module, making M_1 a finitely generated projective R_1 -module, and similarly M_2 is a finitely generated projective R_2 -module. Choosing a basis $\{b_1, \dots, b_k\}$ for $M \oplus P$ over R clearly gives a basis $\{e_1 b_1, \dots, e_1 b_k\}$ for $e_1 M \oplus e_1 P$ over R_1 , and the case of $e_2 M \oplus e_2 P$ is similar. Now if $(r_{i,j}) = (e_1 r_{i,j}) + (e_2 r_{i,j})$ is the matrix of $\phi \oplus 1_P$, then $(e_1 r_{i,j})$ is the matrix of $\phi_1 \oplus 1_{P_1}$ over R_1 , and similarly for $\phi_2 \oplus 1_{P_2}$. Thus

$$\det_R(\phi) = \det(r_{i,j}) = (e_1 + e_2) \det(r_{i,j}) = \det_{R_1}(e_1 r_{i,j}) + \det_{R_2}(e_2 r_{i,j}) = \det_{R_1}(\phi_1)e_1 + \det_{R_2}(\phi_2)e_2.$$

Proposition 3.2.

- a. The following are equivalent:

1. $\ker(f)$ is finite
 2. $\ker(f) = \mu_K$
 3. $\text{coker}(f)$ is finite
 4. $f_{\mathbb{R}}$ is an isomorphism
 5. $R(f) \in \mathbb{R}[G]^*$.
- b. We have the following equalities, the last one requiring that one of the equivalent conditions in (a) hold.

(Note that $\mathbb{C}[G]e_{\psi} = \mathbb{C}e_{\psi} \cong \mathbb{C}$)

$$R(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{\mathbb{R}}^{-1}) = \det_{\mathbb{C}[G]}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}) = \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_{\psi}}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}|_{e_{\psi}\mathbb{C}X_K^S}) = \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_{\psi}}(\lambda_{\mathbb{C}} \circ f_{\mathbb{C}}^{-1}|_{e_{\psi}\mathbb{C}X_K^S})^{-1}$$

- c. When G has exponent 2, we have

$$R(f) = \sum_{\psi \in \hat{G}} \det_{\mathbb{R}[G]e_{\psi}}(f_{\mathbb{R}} \circ \lambda_{\mathbb{R}}^{-1}|_{e_{\psi}\mathbb{R}X_K^S}) = \sum_{\psi \in \hat{G}} \det_{\mathbb{R}[G]e_{\psi}}(\lambda_{\mathbb{R}} \circ f_{\mathbb{R}}^{-1}|_{e_{\psi}\mathbb{R}X_K^S})^{-1}.$$

Proof.

- a. These are clear because μ_K is the torsion subgroup of U_K^S , while U_K^S/μ_K and X_K^S are free abelian groups of the same rank.
- b. This follows from Lemma 3.1.a and 3.1.b
- c. This follows from part b. and Lemma 3.1.a

When G has exponent 2, it remains for us to compute $\det_{\mathbb{R}[G]e_{\psi}}(\lambda_{\mathbb{R}} \circ f_{\mathbb{R}}^{-1}|_{e_{\psi}\mathbb{R}X_K^S})$ for each $\psi \in \hat{G}$. To do this, suppose that E is an intermediate field between F and K , and $H = \text{Gal}(K/E)$. Let $N_H = \sum_{\sigma \in H} \sigma$. For $w \in S_E$, let $\tilde{w} \in S_K$ be a choice of a prime above w in K . There is a natural injective $\mathbb{Z}[G]$ -module map $Y_E \rightarrow Y_K$ which sends each $w \in S_E$

to $N_H \tilde{w}$. We let $\gamma_{K/E} : X_E \rightarrow X_K$ denote the restriction of this map to X_E . Similarly, let $\pi_{K/E}$ be the restriction to X_K of the $\mathbb{Z}[G]$ -module map which sends each prime $\tilde{w} \in S_K$ to the corresponding prime w of E , and note that the image of $\pi_{K/E}$ lies in X_E . It is easy to see that $\gamma_{K/E}$ gives an isomorphism between X_E^S and $N_H(X_K^S)$, and that for $u \in U_E$, we have $\lambda_K(u) = \gamma_{K/E, \mathbb{R}}(\lambda_E(u))$.

Lemma 3.3. *Suppose that $\ker(f)$ is finite. Let $\pi_{G/H} : \mathbb{R}[G] \rightarrow \mathbb{R}[G/H]$ be the natural projection map. If χ is a first degree character of $\overline{G} = G/H$ and $\psi \in \hat{G}$ is its inflation, recall that $r(\chi)$ denotes the dimension of $e_\chi \mathbb{R}X_E$ as a real vector space. Then*

$$\pi_{G/H}(R(f)e_\psi) = |H|^{-r(\chi)} R(\pi_{K/E} \circ f|_{U_E})e_\chi.$$

Proof. (See [6, I.6.4i]) By Proposition 3.2(c),

$$\pi_{G/H}(R(f)^{-1}e_\psi) = \pi_{G/H}\left(\det_{\mathbb{R}[G]e_\psi}(\lambda_{K, \mathbb{R}} \circ f_{\mathbb{R}}^{-1}|_{e_\psi \mathbb{R}X_K})\right).$$

Since $\gamma_{K/E}(X_E) \subset N_H(X_K)$, and $f_{\mathbb{R}}^{-1}$ is an $\mathbb{R}[G]$ -homomorphism, we see that the image of $f_{\mathbb{R}}^{-1} \circ \gamma_{K/E, \mathbb{R}}^S$ is contained in $N_H(\mathbb{R}U_K) \subset \mathbb{R}U_E$. Thus we may follow this map with $\gamma_{K/E, \mathbb{R}} \circ \lambda_{E, \mathbb{R}} = \lambda_{K, \mathbb{R}}|_{\mathbb{R}U_E}$ and obtain

$$\gamma_{K/E, \mathbb{R}} \circ \lambda_{E, \mathbb{R}} \circ f_{\mathbb{R}}^{-1} \circ \gamma_{K/E, \mathbb{R}} = \lambda_{K, \mathbb{R}}^S|_{\mathbb{R}U_E} \circ f_{\mathbb{R}}^{-1} \circ \gamma_{K/E, \mathbb{R}}.$$

Restricting the isomorphism $\gamma_{K/E, \mathbb{R}} : \mathbb{R}X_E \rightarrow N_H(\mathbb{R}X_K)$ gives an isomorphism between $e_\chi \mathbb{R}X_E = e_\psi \mathbb{R}X_E$ and $e_\psi N_H(\mathbb{R}X_K) = |H|e_\psi \mathbb{R}X_K = e_\psi \mathbb{R}X_K$. So, restricting the functions in the last displayed equation to $e_\chi \mathbb{R}X_E$ and noting that $\pi_{G/H}(e_\psi) = e_\chi$, gives

$$\det_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E, \mathbb{R}} \circ (f_{\mathbb{R}}^{-1} \circ \gamma_{K/E, \mathbb{R}})|_{e_\chi \mathbb{R}X_E}) = \pi_{G/H}\left(\det_{\mathbb{R}[G]e_\psi}(\lambda_{K, \mathbb{R}} \circ f_{\mathbb{R}}^{-1}|_{e_\psi \mathbb{R}X_K})\right).$$

Since $\gamma_{K/E, \mathbb{R}}|_{e_\chi \mathbb{R}X_E} : e_\chi \mathbb{R}X_E \rightarrow e_\psi \mathbb{R}X_K$ has the inverse $\frac{1}{|H|}\pi_{K/E, \mathbb{R}}|_{e_\psi \mathbb{R}X_K}$, we get from Proposition 3.2(c) again that

$$\begin{aligned} \det_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E, \mathbb{R}} \circ (f_{\mathbb{R}}^{-1} \circ \gamma_{K/E, \mathbb{R}})|_{e_\chi \mathbb{R}X_E}) &= \det_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E, \mathbb{R}} \circ \left(\frac{1}{|H|}\pi_{K/E} \circ f|_{U_E}\right)^{-1}|_{e_\chi \mathbb{R}X_E}) = \\ &= |H|^{r(\chi)} R(\pi_{K/E} \circ f|_{U_E})^{-1} e_\chi. \end{aligned}$$

Combining the displayed equations gives the result.

Lemma 3.4. *Suppose that E/F is relative quadratic and τ is the non-trivial automorphism of E over F . Let χ be the non-trivial character of $\overline{G} = \text{Gal}(E/F) = \langle \tau \rangle$. If $\overline{f} : U_E \rightarrow X_E$ is a $\mathbb{Z}[\overline{G}]$ -module homomorphism with finite kernel, then*

$$R(\overline{f})e_\chi = ((X_E)_{1+\tau} : \overline{f}((U_E)^{1-\tau}) \frac{R_F w_E}{R_E w_F} \frac{2^{|S|-1-r(\chi)}}{|\mu_E \cap (U_E)^{1-\tau}|}.$$

Proof. Let $M = (X_E : \overline{f}(U_E))$, and let $\overline{f}_0 : U_E/\mu_E \rightarrow f(U)$ be the induced isomorphism.

Then the composite

$$g : X_E \xrightarrow{M} \overline{f}(U_E) \xrightarrow{\overline{f}_0^{-1}} U_E/\mu_E \xrightarrow{w_E} U_E$$

is an injective $\mathbb{Z}[G]$ -module map. For such a map, Tate ([6, I.6.3]) defines $R(\chi, g)$, and it is easy to see that the definition is equivalent to

$$R(\chi, g)e_\chi = \det_{e_\chi \mathbb{R}[G]} (\lambda_{E, \mathbb{R}} \circ g_{\mathbb{R}}|_{e_\chi \mathbb{R}X_E}).$$

By Proposition 3.2(a), $\overline{f}_{\mathbb{R}}$ is an isomorphism, and it is then clear from our definition of g that $g_{\mathbb{R}} = Mw_E \overline{f}_{\mathbb{R}}^{-1}$. Since $r(\chi)$ equals the dimension of $e_\chi \mathbb{R}X_E$ as a real vector space, we see from Proposition 3.2(b) that

$$R(\chi, g)e_\chi = (Mw_E)^{r(\chi)} \det_{\mathbb{R}[G]e_\chi} (\lambda_{E, \mathbb{R}} \circ \overline{f}_{\mathbb{R}}^{-1} \circ |_{e_\chi \mathbb{R}X_E}) = (Mw_E)^{r(\chi)} R(\overline{f})^{-1} e_\chi.$$

On the other hand, the proof of [6, Prop. II.2.1] gives

$$R(g, \chi) = \frac{w_F R_E (U_E^{1-\tau} : g(X_{E_{1+\tau}})^2)}{w_E R_F 2^{|S|-1}}.$$

As an abelian group, $U_E^{1-\tau}$ is the direct product of its torsion subgroup $U_E^{1-\tau} \cap \mu_E$ and a free abelian group of rank $r(\chi)$. Using this and the definition of g , we have

$$(U_E^{1-\tau} : g((X_E)_{1+\tau})^2) = \frac{(U_E^{1-\tau} : (U_E^{1-\tau})^{2Mw_E})}{(g((X_E)_{1+\tau})^2 : (U_E^{1-\tau})^{2Mw_E})} = \frac{|U_E^{1-\tau} \cap \mu_E| (2Mw_E)^{r_s(\chi)}}{(\overline{f}^{-1}(M(X_E)_{1+\tau})^{2w_E} : (U_E^{1-\tau})^{2Mw_E})}.$$

Now $\bar{f}^{-1}(M(X_E)_{1+\tau})^{2w_E}$ is torsion free and hence \bar{f} is injective on this submodule, so we have $\bar{f}^{-1}(M(X_E)_{1+\tau})^{2w_E}/(U_E^{1-\tau})^{2Mw_E} \cong 2Mw_E(X_E)_{1+\tau}/2Mw_E\bar{f}(U_E^{1-\tau})$. Then since X_E is \mathbb{Z} -torsion free,

$$\begin{aligned} (\bar{f}^{-1}(M(X_E)_{1+\tau})^{2w_E} : (U_E^{1-\tau})^{2Mw_E}) = \\ (2Mw_E(X_E)_{1+\tau} : f(2Mw_E U_E^{1-\tau})) = ((X_E)_{1+\tau} : \bar{f}(U_E^{1-\tau})). \end{aligned}$$

Combining the displayed equations gives the result.

Proposition 3.5. *Suppose that $G = \text{Gal}(K/F)$ has exponent 2, ψ is a non-trivial character of G , and $f : U_K \rightarrow X_K$ is a $\mathbb{Z}[G]$ -module homomorphism with finite kernel. Then*

$$R(f)e_\psi = \frac{2^{|S|-1}}{|G|^{r(\psi)}} \frac{w_{E_\psi}}{w_F} \frac{R_F}{R_{E_\psi}} \frac{((X_{E_\psi})_{1+\tau_\psi} : (\pi_{K/E_\psi} \circ f)(U_{E_\psi}^{1-\tau_\psi}))}{|U_{E_\psi}^{1-\tau_\psi} \cap \mu_{E_\psi}|} e_\psi.$$

Proof. Let $E = E_\psi$ and $H = \ker(\psi) = \text{Gal}(K/E)$. Then ψ is the inflation of the nontrivial character χ on $G/H \cong \text{Gal}(E/F) = \bar{G}$. Since $\pi_{G/H}$ restricts to an \mathbb{R} -module isomorphism from $\mathbb{R}[G]e_\psi = \mathbb{R}e_\psi$ to $\mathbb{R}[\bar{G}]e_\chi = \mathbb{R}e_\chi$ with $\pi_{G/H}(e_\psi) = e_\chi$, the result follows directly from Lemmas 3.3 and 3.4.

Lemma 3.6. *For the trivial extension F/F , with identity automorphism σ_0 , and $\bar{f} : U_F \rightarrow X_F$ with finite kernel, we have*

$$R(\bar{f}) = \pm \frac{(X_F : \bar{f}(U_F))}{R_F} \sigma_0.$$

Proof. Let $M = (X_F : \bar{f}(U_F))$, and let $\bar{f}_0 : U_F/\mu_F \rightarrow \bar{f}(U_F)$ be the induced isomorphism. Then the composite

$$g : X_F \xrightarrow{M} \bar{f}(U_F) \xrightarrow{\bar{f}_0^{-1}} U_F/\mu_F \xrightarrow{w_F} U_F$$

is an injective \mathbb{Z} -module map. Then as in the proof of Lemma 3.4,

$$R(1, g) = \det_{\mathbb{R}}(\lambda_{F, \mathbb{R}} \circ g_{\mathbb{R}}) = (Mw_F)^{|S|-1} \det_{\mathbb{R}}(\lambda_{F, \mathbb{R}} \circ \bar{f}_0^{-1}) = (Mw_F)^{|S|-1} R(\bar{f})^{-1}.$$

On the other hand, the proof of [6, Prop. II.1.1] gives

$$R(1, g) = \pm \frac{R_F}{w_F}(U_F : g(X_F)).$$

As an abelian group, U_F is the direct product of its torsion subgroup μ_F and a free abelian group of rank $|S| - 1$. Using this and the definition of g , we have

$$(U_F : g(X_F)) = \frac{(U_F : (U_F)^{Mw_F})}{(g(X_F) : (U_F)^{Mw_F})} = \frac{w_F(Mw_F)^{|S|-1}}{(\bar{f}^{-1}(MX_F)^{w_F} : (U_F)^{Mw_F})}.$$

Now $\bar{f}^{-1}(MX_F)^{w_F}$ is \mathbb{Z} -torsion free and hence \bar{f} is injective on this submodule, so we have $\bar{f}^{-1}(MX_F)^{w_F}/(U_F)^{Mw_F} \cong Mw_F(X_F)/Mw_F\bar{f}(U_F)$. Then since X_F is \mathbb{Z} -torsion free,

$$(\bar{f}^{-1}(MX_F)^{w_F} : (U_F)^{Mw_F}) = (Mw_F X_F : Mw_F \bar{f}(U_F)) = ((X_F) : \bar{f}(U_F)).$$

Combining the displayed equations gives the result.

Proposition 3.7. *Suppose that $G = \text{Gal}(K/F)$ has exponent 2, ψ_0 is the trivial character of G , and $f : U_K \rightarrow X_K$ is a $\mathbb{Z}[G]$ -module homomorphism with finite kernel. Then*

$$R(f)e_{\psi_0} = \frac{(X_F : \pi_{K/F} \circ f(U_F))}{|G|^{|S|-1} R_F} e_{\psi_0}.$$

Proof. Since ψ_0 is the inflation of the trivial character χ_0 on $\text{Gal}(F/F)$, and $\pi_{G/G}$ restricts to an \mathbb{R} -module isomorphism from $\mathbb{R}[G]e_{\psi_0} = \mathbb{R}e_{\psi_0}$ to $\mathbb{R}\sigma_0$ with $\pi_{G/G}(e_{\psi_0}) = \sigma_0$, the result follows from Lemmas 3.3 and 3.6.

IV. CLASS GROUP ANNIHILATORS

Proposition 4.1. *Suppose that $G = \text{Gal}(K/F)$ has exponent 2 and that $f : U_K \rightarrow X_K$ is a $\mathbb{Z}[G]$ -module homomorphism with finite kernel. Then*

$$R(f)\theta_{K/F}^* = \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F |G|^{|S|-1}} e_{\psi_0} + \sum_{\psi \neq \psi_0} \frac{2^{|S|-1} |C_{E_\psi/F}|}{|G|^{r^S(\psi)}} \frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|} e_\psi$$

Proof. Combining Propositions 2.1 and 3.7 gives the coefficient of e_{ψ_0} . Using Propositions 2.2 and 3.5 for $\psi \neq \psi_0$ yields

$$R(f)\theta_{K/F}^* e_\psi = \frac{2^{|S|-1} |C_{E_\psi/F}|}{|G|^{r^S(\psi)} |U_{E_\psi}^{1-\tau_\psi} \cap \mu_{E_\psi}|} \frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi})))}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})} e_\psi.$$

Then

$$\begin{aligned} & ((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))) = \\ & ((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi}))) (\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})) : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))). \end{aligned}$$

Now consider the kernel of $\pi_{K/E_\psi} \circ f$ restricted to U_{E_ψ} . So let $u \in U_{E_\psi}$ and $f(u) = \sum_{w \in S_K} n_w w$. Since $\sigma(u) = u$ for $\sigma \in H = \text{Gal}(K/E_\psi)$, we have $n_w = n_{\sigma(w)}$ for each w . Fix a set of representatives $\{w_i\}$, one for each distinct orbit of S_K under the action of H , and write $w_i \sim w$ if w_i and w lie in the same orbit with cardinality d_i . Then

$$f(u) = \sum_i \sum_{w \sim w_i} n_w w = \sum_i \sum_{w \sim w_i} n_{w_i} w = \sum_i n_{w_i} \sum_{w \sim w_i} w$$

and

$$\pi_{K/E_\psi}(f(u)) = \sum_i n_{w_i} \sum_{w \sim w_i} \pi_{K/E_\psi}(w) = \sum_i n_{w_i} \sum_{w \sim w_i} \pi_{K/E_\psi}(w_i) = \sum_i n_{w_i} d_i \pi_{K/E_\psi}(w_i).$$

Since the elements $\pi_{K/E_\psi}(w_i)$ are distinct, this is zero if and only if each n_{w_i} is zero and hence $f(u) = 0$. Our assumption on f implies that this holds if and only if $u \in \mu_K$. So the kernel of $\pi_{K/E_\psi} \circ f$ restricted to U_{E_ψ} is clearly μ_{E_ψ} . Thus $\pi_{K/E_\psi} \circ f$ induces a homomorphism from $((U_{E_\psi}^S)_{1+\tau_\psi} / (U_{E_\psi}^S)^{1-\tau_\psi})$ onto $\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})) / \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))$ with kernel $(U_{E_\psi}^S)^{1-\tau_\psi} \cap (\mu_{E_\psi})_{1+\tau_\psi}$. Consequently

$$\frac{((U_{E_\psi}^S)_{1+\tau_\psi} : (U_{E_\psi}^S)^{1-\tau_\psi})}{|(U_{E_\psi}^S)^{1-\tau_\psi} \cap (\mu_{E_\psi})_{1+\tau_\psi}|} = (\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})) : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))).$$

Combining the displayed equations then gives the result.

Lemma 4.2. *Suppose that $\alpha \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and that G is the direct product of its subgroups H and J . Let M be the fixed field of H , and identify J with $\text{Gal}(M/F)$ by restriction. Then $\alpha N_H = \beta N_H$, for some $\beta \in \text{Ann}_{\mathbb{Z}[J]}(\mu_M)$.*

Proof. Write

$$\alpha = \sum_{\rho \in J} \sum_{\sigma \in H} n_{\rho\sigma} \rho\sigma \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K).$$

Restricting to M , we define

$$\beta = \sum_{\rho \in J} \left(\sum_{\sigma \in H} n_{\rho\sigma} \right) \rho \in \text{Ann}_{\mathbb{Z}[J]}(\mu_M).$$

Note that

$$(\alpha - \beta) = \sum_{\rho \in J} \sum_{\sigma \in H} n_{\rho\sigma} \rho(\sigma - 1).$$

Since $(\sigma - 1)N_H = 0$ for each $\sigma \in H$, we have $(\alpha - \beta)N_H = 0$ and thus $\alpha N_H = \beta N_H$, as desired.

Corollary 4.3. *Suppose that $\alpha \in \text{Ann}_{\mathbb{Z}[G]}(\mu_K)$.*

1. *Then $\alpha N_G = c w_F N_G$ for some $c \in \mathbb{Z}$.*
2. *Suppose that E is a quadratic extension of F in K , with $H = \text{Gal}(K/E)$, and $H \not\subset J = \langle \tau \rangle$ of order 2, so that G is the direct product of H and J . Then $\alpha N_H(1 - \tau) = d |(\mu_E)_{1+\tau}| N_H(1 - \tau)$ for some integer d .*

Proof.

1. Applying Lemma 4.2 with $H = G$ and J trivial gives $\alpha N_G = \beta N_G$ with $\beta \in \text{Ann}_{\mathbb{Z}}(\mu_F) = w_F \mathbb{Z}$. So $\beta = c w_F$, giving the desired result.
2. First applying Lemma 4.2 with $M = E$ gives

$$\alpha N_H = \beta N_H$$

with $\beta \in \text{Ann}_{\mathbb{Z}[J]}(\mu_E)$. Now $\mathbb{Z}[J] = \mathbb{Z} + \mathbb{Z}\tau$, so $\beta = m + n\tau$ with $m, n \in \mathbb{Z}$. Since β annihilates $(\mu_E)_{1+\tau}$ on which τ acts as -1 , we have

$$1 = ((\mu_E)_{1+\tau})^\beta = ((\mu_E)_{1+\tau})^{m+n\tau} = ((\mu_E)_{1+\tau})^{m-n}.$$

Therefore $m - n \in \text{Ann}_{\mathbb{Z}}((\mu_E)_{1+\tau}) = |(\mu_E)_{1+\tau}|\mathbb{Z}$, and $m - n = d|(\mu_E)_{1+\tau}|$. Finally,

$$\beta(1 - \tau) = (m + n\tau)(1 - \tau) = (m - n)(1 - \tau) = d|(\mu_E)_{1+\tau}|(1 - \tau).$$

Combining this with the first displayed equation gives the result.

Proposition 4.4. *If $\psi \neq \psi_0$ and the integer b is an exponent for $C_{E_\psi/F}$, then $b|G|e_\psi$ annihilates Cl_K^S . Indeed, if \mathfrak{a} is an ideal of \mathcal{O}_K^S , then $\mathfrak{a}^{b|G|e_\psi} = \delta\mathcal{O}_K^S$ for some $\delta \in (E_\psi)_{1+\tau_\psi}$.*

Proof. Let $H = \text{Gal}(K/E_\psi)$ and let τ_ψ be a fixed lift of a generator of $\text{Gal}(E_\psi/F)$ to G . Then

$$b|G|e_\psi = bN_H(1 - \tau_\psi).$$

Any element of Cl_K^S is represented by an ideal \mathfrak{a}_K of \mathcal{O}_K^S . Then

$$\mathfrak{a}_K^{N_H} = \mathfrak{a}_E\mathcal{O}_K^S$$

for some ideal \mathfrak{a}_E of $\mathcal{O}_{E_\psi}^S$, while

$$\mathfrak{a}_E^b = \gamma\mathfrak{a}_F\mathcal{O}_{E_\psi}^S$$

for some ideal \mathfrak{a}_F of \mathcal{O}_F^S and $0 \neq \gamma \in E_\psi$, since b annihilates $\text{Cl}_{E_\psi}^S$ modulo the image of Cl_F^S . Finally,

$$(\gamma\mathfrak{a}_F)^{1-\tau_\psi} = \gamma^{1-\tau_\psi}\mathfrak{a}_F^{1-\tau_\psi} = \gamma^{1-\tau_\psi}\mathcal{O}_E^S.$$

Since $\delta = \gamma^{1-\tau_\psi} \in (E_\psi)_{1+\tau_\psi}$, combining the displayed equations gives the result.

Theorem 4.5. *Let K be a composite of a finite number of quadratic extensions of a number field F . Let S contain the infinite primes of F and those which ramify in K/F . Suppose that $\ker(f)$ is finite and $\alpha \in \mathbb{Z}[G]$ annihilates μ_K . Let ψ be an irreducible character of G . Then $|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ lies in $\mathbb{Z}[G]$ and annihilates Cl_K^S . Indeed, if \mathfrak{a} is an ideal of \mathcal{O}_K^S , then $\mathfrak{a}^{|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi} = \delta \mathcal{O}_K^S$ for some $\delta \in F$ when $\psi = \psi_0$, and for some δ satisfying $\delta \in (E_\psi)_{1+\tau_\psi}$ when $\psi \neq \psi_0$.*

Proof. First consider $\psi = \psi_0$. Note that $|G|e_{\psi_0} = N_G$ and $r^S(\psi_0) = |S| - 1$. Using Proposition 4.1 and Corollary 4.3.1 yields

$$\begin{aligned} |G|^{r^S(\psi_0)+1}\alpha R(f)\theta_{K/F}^{S,*}e_{\psi_0} &= |G|^{|S|-1}R(f)\theta_{K/F}^{S,*}e_{\psi_0}\alpha|G|e_{\psi_0} = \\ &= \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F}\alpha N_G = \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F}c w_F N_G = \\ &= h_F(X_F : \pi_{K/F}(f(U_F)))c N_G, \end{aligned}$$

which clearly lies in $\mathbb{Z}[G]$. Now any element of Cl_K^S is represented by an ideal \mathfrak{a}_K of \mathcal{O}_K^S , and

$$\mathfrak{a}_K^{N_G} = \mathfrak{a}_F \mathcal{O}_K^S$$

for some ideal \mathfrak{a}_F of \mathcal{O}_F^S . Then

$$\mathfrak{a}_F^{h_f} = \gamma \mathcal{O}_F^S,$$

for some $\gamma \in F$. Thus the result follows from the displayed equations, with $\delta = \gamma^{(X_F : \pi_{K/F}(f(U_F)))c}$.

Next consider $\psi \neq \psi_0$. Put $H = \text{Gal}(K/E_\psi)$ and let τ_ψ be a fixed lift of a generator of $\text{Gal}(E_\psi/F)$ to G . Then $|G|e_\psi = N_H(1 - \tau_\psi)$. Using Proposition 4.1 and Corollary 4.3.2

yields

$$\begin{aligned}
|G|^{r^S(\psi)+1} \alpha R(f) \theta_{K/F}^{S,*} e_\psi &= |G|^{r^S(\psi)} R(f) \theta_{K/F}^{S,*} e_\psi \alpha |G| e_\psi = \\
2^{|S|-1} |C_{E_\psi/F}| &\frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|} e_\psi \alpha |G| e_\psi = \\
2^{|S|-1} |C_{E_\psi/F}| &\frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|} e_\psi d |G| e_\psi = \\
2^{|S|-1} |C_{E_\psi/F}| &((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi}))) d |G| e_\psi
\end{aligned}$$

Since this is an integer multiple of $|C_{E_\psi/F}| |G| e_\psi = |C_{E_\psi/F}| N_H(1 - \tau_\psi)$, the result follows from Proposition 4.4.

Remark 4.6. *It is clear from the proof of Theorem 4.5 that in fact $\frac{|G|^{r^S(\psi)+1}}{2^{|S|-1}} \alpha R(f) \theta_{K/F}^{S,*} e_\psi$ annihilates Cl_K^S when $\psi \neq \psi_0$. Furthermore, in this situation, if r_F^S denotes the 2-rank of Cl_F^S , one can show by an argument similar to that in [3, Proposition 2], that the 2-rank of $C_{E_\psi/F}$ is always at least $r_F^S - 1$. Thus $|C_{E_\psi/F}| / (2^{r_F^S - 2})$ suffices as an exponent for $C_{E_\psi/F}$, and this allows one to modify the proof of Theorem 4.5 to conclude that $\frac{|G|^{r^S(\psi)+1}}{2^{|S|+r_F^S-3}} \alpha R(f) \theta_{K/F}^{S,*} e_\psi$ annihilates Cl_K^S when $\psi \neq \psi_0$, and that $\frac{|G|^{r^S(\psi)+1}}{2^{r_F^S-1}} \alpha R(f) \theta_{K/F}^{S,*} e_\psi$ does so when $\psi = \psi_0$. Finally, [3, Cor. 2] shows that $2^{|S|+r_F^S}$ is an integer multiple of $|G|$, so that for $\psi \neq \psi_0$, we see that $2^3 |G|^{r^S(\psi)} \alpha R(f) \theta_{K/F}^{S,*} e_\psi$ annihilates Cl_F^S .*

Remark 4.7. *By analogy with the Brumer-Stark conjecture, one may also be interested in further properties of the generator δ in Theorem 4.5. The conditions given there guarantee that $K(\sqrt{\delta})/F$ is an abelian Galois extension in all cases. If F has a real embedding, and $\psi \neq \psi_0$, the condition $\delta \in (E_\psi)_{1+\tau_\psi}$ suffices to imply that $K(\delta^{1/w_{E_\psi}})/F$ is an abelian Galois extension, by application of [6, Proposition IV.1.2]. Indeed, $E_\psi(\delta^{1/w_{E_\psi}})/F$ is abelian by the criterion there since $1 + \tau_\psi$ annihilates μ_{E_ψ} in this case and $\delta^{1+\tau_\psi} = 1$ which is a w_{E_ψ} -power.*

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