

## **L-VALUES FOR BIQUADRATIC EXTENSIONS AND THE FITTING IDEAL OF THE TAME KERNEL**

JONATHAN W. SANDS

*Dedicated to Paulo Ribenboim on the occasion of his 80th birthday.*

RÉSUMÉ. Soit  $\mathcal{E}/F$  une extension de Galois totalement réelle de corps de nombres telle que le groupe de Galois  $G$  est isomorphe au groupe de Klein d'ordre 4 et supposons que la conjecture de Birch-Tate est vérifiée pour tous les corps compris entre  $F$  et  $\mathcal{E}$ , inclusivement. Soit  $S$  un ensemble fini de premiers de  $F$  contenant les premiers infinis et tous ceux qui se ramifient dans  $\mathcal{E}$ , soit  $S_{\mathcal{E}}$  l'ensemble des premiers de  $\mathcal{E}$  au-dessus de ceux de  $S$ , et soit  $\mathcal{O}_{\mathcal{E}}^S$  l'anneau des  $S_{\mathcal{E}}$ -entiers de  $\mathcal{E}$ . Lorsque  $\mathcal{E}$  peut être judicieusement plongé dans des extensions diédrales de  $F$ , nous montrons que l'idéal de Fitting de  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  et un certain idéal de Stickelberger supérieur dans  $\mathbb{Z}[G]$  sont tous les deux d'indice un ou deux dans leur somme.

ABSTRACT. Fix a Galois extension  $\mathcal{E}/F$  of totally real number fields such that the Galois group  $G$  is isomorphic to the Klein four group and assume that the Birch-Tate conjecture holds in all the intermediate fields between  $F$  and  $\mathcal{E}$ , inclusive. Let  $S$  be a finite set of primes of  $F$  containing the infinite primes and all those which ramify in  $\mathcal{E}$ , let  $S_{\mathcal{E}}$  denote the primes of  $\mathcal{E}$  lying above those in  $S$ , and let  $\mathcal{O}_{\mathcal{E}}^S$  denote the ring of  $S_{\mathcal{E}}$ -integers of  $\mathcal{E}$ . When  $\mathcal{E}$  can be embedded in dihedral extensions of  $F$  in certain ways, we show that the Fitting ideal of  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  and a higher Stickelberger ideal in  $\mathbb{Z}[G]$  both have index one or two in their sum.

### **1. Introduction**

Fix an abelian Galois extension of number fields  $\mathcal{E}/F$  and let  $G$  denote the Galois group. Also fix a finite set  $S$  of primes of  $F$  which contains all of the infinite primes of  $F$  and all of the primes which ramify in  $\mathcal{E}$ . Associated with this data is a Stickelberger function,  $\theta_{\mathcal{E}/F}^S(s)$ , a meromorphic function of  $s$  with values in the group ring  $\mathbb{C}[G]$ . It can be defined when the real part of  $s$  is greater than 1, as a product over the (finite) primes  $\mathfrak{p}$  of  $F$  that are not in  $S$ . Let  $N\mathfrak{p}$  denote the absolute norm of the ideal  $\mathfrak{p}$  and  $\sigma_{\mathfrak{p}} \in G$  denote the Frobenius automorphism of  $\mathfrak{p}$ . Then

$$\theta_{\mathcal{E}/F}^S(s) = \prod_{\text{prime } \mathfrak{p} \notin S} \left( 1 - \frac{1}{N\mathfrak{p}^s} \sigma_{\mathfrak{p}}^{-1} \right)^{-1}.$$

This extends meromorphically to all of  $\mathbb{C}$ . When  $\mathcal{E} = F$ , the function  $\theta_{F/F}^S(s)$  is simply the identity automorphism of  $F$  times  $\zeta_F^S(s)$ , the Dedekind zeta-function of  $F$  with Euler factors for the primes in  $S$  removed.

The function  $\theta_{\mathcal{E}/F}^S(s)$  is connected with the arithmetic of the number fields  $\mathcal{E}$  and  $F$  in ways one would like to make as precise as possible. The ring of  $S$ -integers  $\mathcal{O}_F^S$  of  $F$  is defined to be the set of elements of  $F$  whose valuation is non-negative at every prime not in  $S$ . Similarly, define the ring  $\mathcal{O}_{\mathcal{E}}^S$  of  $S$ -integers of  $\mathcal{E}$  to be the set of elements of  $\mathcal{E}$  whose valuation is non-negative at every prime not in  $S_{\mathcal{E}}$ , the set of all primes of  $\mathcal{E}$  which lie above some prime in  $S$ . The function  $\zeta_F^S(s)$  may be viewed as the zeta-function of the Dedekind domain  $\mathcal{O}_F^S$ .

We will study the “higher Stickelberger element”  $\theta_{\mathcal{E}/F}^S(-1)$ , which lies in  $\mathbb{Q}[G]$  by the theorem of Klingen-Siegel [11], and is related to the algebraic  $K$ -group  $K_2(\mathcal{O}_{\mathcal{E}}^S)$ . Denoting the valuation at a finite prime  $\mathfrak{p}$  of  $\mathcal{E}$  by  $v_{\mathfrak{p}}$ , the group  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  may be described as the subgroup of  $K_2(\mathcal{E})$  consisting of all elements whose image under the map which sends  $\{\gamma, \alpha\}_{\mathcal{E}}$  to the tame symbol

$$(\gamma, \alpha)_{\mathfrak{p}} = -1^{v_{\mathfrak{p}}(\gamma)v_{\mathfrak{p}}(\alpha)} \gamma^{v_{\mathfrak{p}}(\alpha)} / \alpha^{v_{\mathfrak{p}}(\gamma)} \pmod{\mathfrak{p}}$$

is trivial in the residue field modulo  $\mathfrak{p}$  for every prime  $\mathfrak{p}$  not in  $S_{\mathcal{E}}$ . This group  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  is known to be finite by [2] and [7], and will be called the *S-tame kernel* of  $\mathcal{E}$ . It contains the tame kernel  $K_2(\mathcal{O}_{\mathcal{E}})$  as a subgroup.

Another piece of the arithmetic interpretation of  $\theta_{\mathcal{E}/F}^S(-1)$  involves a group of roots of unity. Let  $\mu_{\infty}$  denote the group of all roots of unity in an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$  containing  $\mathcal{E}$ , and let  $\mathcal{G}$  denote the Galois group of  $\overline{\mathbb{Q}}/\mathbb{Q}$ . Define  $W_2 = W_2(\overline{\mathbb{Q}})$  to be the  $\mathbb{Z}[\mathcal{G}]$ -module whose underlying group is  $\mu_{\infty}$ , with the action of  $\gamma \in \mathcal{G}$  on  $\omega \in W_2$  given by  $\omega^{\gamma} = \gamma^2(\omega)$ . For any subfield  $L$  of  $\overline{\mathbb{Q}}$ , let  $W_2(L)$  be the submodule fixed under this action by the Galois group of  $\overline{\mathbb{Q}}$  over  $L$ . Then  $W_2(\mathcal{E})$  naturally becomes a  $\mathbb{Z}[G]$ -module, where the action of  $G$  arises by lifting elements of  $G$  to  $\mathcal{G}$  and then using the action of  $\mathcal{G}$  just defined. One easily sees that the  $G$ -fixed submodule  $W_2(\mathcal{E})^G$  equals  $W_2(F)$ . We use the notation  $w_2(L) = |W_2(L)|$ , which we note is finite for any algebraic number field  $L$ .

Our approach makes use of the conjecture of Birch and Tate (see Section 4 in [12]), which gives a precise arithmetic interpretation of  $\zeta_F^S(-1)$ . We state a form of it for an arbitrary finite set  $S$  which is easily seen to be equivalent to the original conjecture for the set  $S$  containing just the infinite primes (see Corollary 3.3 of [10]).

**Conjecture 1.1 (Birch-Tate).** *Suppose that  $F$  is totally real and that the finite set  $S$  contains the infinite primes of  $F$ . Then*

$$\zeta_F^S(-1) = (-1)^{|S|} \frac{|K_2(\mathcal{O}_F^S)|}{w_2(F)}.$$

Results on Iwasawa’s main conjecture in [6] and [14] lead to the following (see [4]).

**Proposition 1.2.** *The Birch-Tate conjecture holds if  $F$  is abelian over  $\mathbb{Q}$ , and the odd part holds for all totally real  $F$ .*

Kolster [3] has shown that the 2-part of the Birch-Tate conjecture for  $F$  would follow from the 2-part of Iwasawa's Main conjecture for  $F$ .

For any module  $M$  over a commutative ring  $A$ , we let  $\text{Ann}_A(M)$  denote the annihilator of  $M$  in  $A$ . If  $M$  is a finitely generated  $A$ -module, we denote the *Fitting ideal* of  $M$  over  $A$  by  $\text{Fit}_A(M)$ ; it is the ideal of  $A$  generated by the determinants of all square matrices representing relations among a set of generators of  $A$ . The following result is proved in Theorem 1.3 of [8]. The notation  $\overline{G}$  for the Galois group reflects that in later applications this group will in fact be a homomorphic image of a group  $G$ .

**Proposition 1.3.** *Let  $E/F$  be a relative quadratic extension of totally real number fields, with Galois group  $\overline{G}$ . Let  $S$  contain the infinite primes and those which ramify in  $E/F$ . Assume that the 2-part of the Birch-Tate conjecture holds for  $E$  and for  $F$ . Then the (first) Fitting ideal of  $K_2(\mathcal{O}_E^S)$  as a  $\mathbb{Z}[\overline{G}]$ -module is*

$$\text{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S)) = \text{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1).$$

More specifically, this ideal equals its extension to the maximal order of  $\mathbb{Q}[\overline{G}]$  if and only if it is not principal, and this happens exactly when  $E$  is not the first layer of the cyclotomic  $\mathbb{Z}_2$ -extension of  $F$ . Without the assumption of the Birch-Tate conjecture, the ideals  $\text{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S))$  and  $\text{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1)$  have the same extension to  $\mathbb{Z}[1/2][\overline{G}]$ .

In this paper, we investigate the relationship between the Fitting ideal

$$\text{Fit}_{\mathcal{E}/F}^S(1) = \text{Fit}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_{\mathcal{E}}^S))$$

and the corresponding higher Stickelberger ideal

$$\text{Stick}_{\mathcal{E}/F}^S(-1) = \text{Ann}_{\mathbb{Z}[G]}(W_2(\mathcal{E}))\theta_{\mathcal{E}/F}^S(-1)$$

when  $G$  is the Klein four group. The theorem of Deligne and Ribet [1] guarantees that  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  is an ideal in the integral group ring  $\mathbb{Z}[G]$ .

## 2. Biquadratic extensions

From now on, we let  $\mathcal{E}/F$  be a biquadratic extension of totally real number fields with intermediate fields  $E_1, E_2$  and  $E_3$  and assume that  $\mathcal{E}$  is contained in  $\mathbb{R}$ .

One particular quadratic extension of  $F$  plays a special role, namely the first layer  $F^{(1)}$  of the cyclotomic  $\mathbb{Z}_2$ -extension of  $F$ . This extension may be described by setting  $\zeta_k = e^{2\pi i/k}$  and choosing the largest positive integer  $n$  such that  $\pi_F = 2 + \zeta_{2^n} + \zeta_{2^n}^{-1}$  lies in  $F$ . Then  $F^{(1)} = F(\sqrt{\pi_F})$ , and  $\sqrt{\pi_F} = \zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1}$ , whose absolute norm is a power of 2.

The following two results appear in Theorem 4.5, Theorem 4.8 and Proposition 4.9 in [9]. Our goal will be to obtain new general applications of these. Here we let  $e_0$  be the idempotent in  $\mathbb{Q}[G]$  corresponding to the trivial character of  $G$ , and let  $e_i$  be the idempotent corresponding to the non-trivial character whose kernel fixes  $E_i$ . Then  $\mathcal{S} = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$  is the maximal order of  $\mathbb{Q}[G]$ . It is easy to compute that  $4\mathcal{S}$  has index 16 in  $\mathbb{Z}[G]$ , and hence  $\mathbb{Z}[G]$  has index 16 in  $\mathcal{S}$ . The symbol  $k_2^S(F)$  will

denote the order of  $K_2(\mathcal{O}_F^S)$ , and  $k_2^S(E_i)^-$  will denote the order of the submodule of elements in  $K_2(\mathcal{O}_{E_i}^S)$  that are inverted by the non-trivial automorphism of  $E_i$  over  $F$ .

**Theorem 2.1 (Comparison Theorem for a biquadratic extension not containing  $F^{(1)}$ ).** *Suppose that  $\mathcal{E}/F$  is biquadratic, and that  $\mathcal{E}$  does not contain  $F^{(1)}$ . If the intersection of the images in  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  of  $K_2(\mathcal{O}_{E_1}^S)$  and  $K_2(\mathcal{O}_{E_3}^S)$  does not equal the image of  $K_2(\mathcal{O}_F^S)$ , and likewise for the intersection of the images in  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  of  $K_2(\mathcal{O}_{E_1}^S)$  and  $K_2(\mathcal{O}_{E_2}^S)$ , then either*

- (a)  $\text{Fit}_{\mathcal{E}/F}^S(1) = \text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}$ , or
- (b)  $\text{Fit}_{\mathcal{E}/F}^S(1)$  has index 2 in  $\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}$ .

Suppose further that the Birch-Tate conjecture holds for  $F$  and the  $E_i$ 's. Then

$$\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S} = \text{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S},$$

which contains  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  of index 2. In case (a),  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  is contained in  $\text{Fit}_{\mathcal{E}/F}^S(1)$  with index 2. In case (b),  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  and  $\text{Fit}_{\mathcal{E}/F}^S(1)$  both have index 2 in  $\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S} = \text{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S}$ , and thus have the same index in  $\mathbb{Z}[G]$ . Assuming that the Birch-Tate conjecture also holds for  $\mathcal{E}$ , the index of  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  in  $\mathbb{Z}[G]$  equals the order of  $K_2(\mathcal{O}_E^S)$  in both cases.

**Theorem 2.2 (Comparison Theorem for a biquadratic extension containing  $F^{(1)}$ ).** *Suppose that  $\mathcal{E}/F$  is biquadratic, and that  $E_1 = F^{(1)}$ .*

- (i) Then

$$\text{Fit}_{\mathcal{E}/F}^S(1) \supset 2\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}$$

and  $\frac{\text{Fit}_{\mathcal{E}/F}^S(1)}{2\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}}$  must be one of three  $\mathbb{F}_2$ -subspaces of  $\frac{\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}}{2\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S}}$ . (We cannot say that all three occur.) The bases for these subspaces are:

- (a)  $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1, k_2^S(E_2)^-e_2, k_2^S(E_3)^-e_3\}$ ,
- (b)  $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1 + k_2^S(E_2)^-e_2, k_2^S(E_2)^-e_2 + k_2^S(E_3)^-e_3\}$ ,
- (c)  $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1, k_2^S(E_2)^-e_2 + k_2^S(E_3)^-e_3\}$ .

- (ii) Now assume that the Birch-Tate conjecture holds for  $F$  and each  $E_i$ 's. Then

$$\text{Fit}_{\mathcal{E}/F}^S(1)\mathcal{S} = \text{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S},$$

which contains  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  of index 4. If case (a) occurs,  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  lies in  $\text{Fit}_{\mathcal{E}/F}^S(1)$  with index 2. If case (b) occurs,  $\text{Fit}_{\mathcal{E}/F}^S(1) = \text{Stick}_{\mathcal{E}/F}^S(-1)$ . If case (c) occurs,  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  and  $\text{Fit}_{\mathcal{E}/F}^S(1)$  have the same index in  $\mathbb{Z}[G]$ . If the intersection of the images in  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  of  $K_2(\mathcal{O}_{E_1}^S)$  and  $K_2(\mathcal{O}_{E_3}^S)$  does not equal the image of  $K_2(\mathcal{O}_F^S)$ , then case (c) does not occur.

(iii) If the Birch-Tate conjecture holds for  $\mathcal{E}$ , then the index of  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  in  $\mathbb{Z}[G]$  equals the order of  $K_2(\mathcal{O}_E^S)$  in all cases.

**Proposition 2.3.** *Suppose that  $F$  is totally real and  $E = F(\sqrt{d})$  for some non-zero, totally positive  $d \in F$ . Then the kernel of the natural map  $\iota_{E/F} : K_2(F) \rightarrow K_2(E)$  is generated by the symbol  $\{-1, d\}_F$ .*

**Proof.** Suppose that  $\omega$  is in the kernel. So  $\iota_{E/F}(\omega) = 1$ . Applying the transfer  $\text{Tr}_{E/F}$  shows that  $\omega^2 = 1$ . Then by Theorem 6.1 in [13],  $\omega = \{-1, a\}_F$  for some non-zero  $a \in F$ . Hence our assumption is that  $\{-1, a\}_E = 1$ . Thus  $a \in F$  lies in the Tate kernel of  $E$ . Since  $E$  is totally real, the Tate kernel of  $E$  is generated by the squares in  $E^\times$  and  $\pi_E$  for which  $E(\sqrt{\pi_E})$  is the first layer  $E^{(1)}$  of the cyclotomic  $\mathbb{Z}_2$  extension of  $E$  (see Proposition 2.4 in [5]). Thus  $F(\sqrt{a}) \subset E^{(1)} = F(\sqrt{d}, \sqrt{\pi_E})$ .

Firstly, if  $E = F^{(1)}$ , then  $E^{(1)}/F$  is cyclic, being contained in the cyclotomic  $\mathbb{Z}_2$ -extension of  $F$ . So  $F(\sqrt{a}) \subset F(\sqrt{d}) = E = F^{(1)} = F(\sqrt{\pi_F})$ , the only quadratic extension of  $F$  in  $E^{(1)}$ . By Kummer theory,  $a$  and  $d$  must both equal a power of  $\pi_F$  times a square in  $F^\times$ . Thus they both lie in the Tate kernel of  $F$ , and

$$\omega = \{-1, a\}_F = 1 = \{-1, d\}_F.$$

So the kernel of  $\iota_{E/F}$  is trivial and we are done in this case.

Secondly, if  $E \neq F^{(1)}$ , then we have  $\pi_E = \pi_F \in F$ . This time, Kummer theory implies that  $a$  lies in the subgroup of  $F^\times$  generated by the squares, along with  $d$  and  $\pi_F$ . Since  $\{-1, \pi_F\}_F = 1$ , we see that  $\{-1, a\}_F$  is a power of  $\{-1, d\}_F$ .  $\square$

**Proposition 2.4.** *Let  $F$  be a totally real algebraic number field. Let  $E_1 = F(\sqrt{d_1})$  and  $E_3 = F(\sqrt{d_3})$  be distinct totally real quadratic extensions of  $F$  with composite  $\mathcal{E}$ , and assume that  $E_2 = F(\sqrt{d_1 d_3}) \neq F^{(1)}$ . If  $\{-1, \alpha_1\}_\mathcal{E} = \{-1, \alpha_3\}_\mathcal{E} = \iota_{\mathcal{E}/F}(\omega)$ , for some  $\alpha_1 \in E_1, \alpha_3 \in E_3$  and  $\omega \in K_2(F)$ , then  $\omega^2 = 1$ .*

**Proof.** Applying the transfer map  $\text{Tr}_{\mathcal{E}/E_1}$  and using its standard properties gives

$$\begin{aligned} 1 &= \{1, \alpha_1\}_{E_1} = \{-1, \alpha_1\}_{E_1}^2 = \{-1, \alpha_1^2\}_{E_1} = \{-1, N_{\mathcal{E}/E_1}(\alpha_1)\}_{E_1} \\ &= \text{Tr}_{\mathcal{E}/E_1}(\{-1, \alpha_1\}_\mathcal{E}) = \text{Tr}_{\mathcal{E}/E_1}(\iota_{\mathcal{E}/E_1}(\iota_{E_1/F}(\omega))) = \iota_{E_1/F}(\omega)^2 = \iota_{E_1/F}(\omega^2). \end{aligned}$$

Thus  $\omega^2$  lies in the kernel of  $\iota_{E_1/F}$ . By Proposition 2.3,  $\omega^2$  is a power of  $\{-1, d_1\}_F$ . The proof of Proposition 2.3 then also shows that  $\omega^2 = 1$  if  $E_1 = F^{(1)}$ , and we are done in this case. By the same argument,  $\omega^2$  is a power of  $\{-1, d_3\}_F$ , and we are done if  $E_3 = F^{(1)}$ . In the remaining case, assume by way of contradiction that  $\omega^2 \neq 1$ . Then we must have  $\{-1, d_1\}_F = \omega^2 = \{-1, d_3\}_F$ , so that  $d_1 d_3$  is in the Tate kernel of  $F$ . As in the proof of Proposition 2.3, this implies that  $E_2 = F(\sqrt{d_1 d_3})$  lies in  $F^{(1)}$ , contradicting our assumptions.  $\square$

**Lemma 2.5.** *Let  $\mathcal{E}$  be a biquadratic extension of  $F$ , with intermediate subfields  $E_1, E_2$  and  $E_3$ . If  $M$  is an extension of  $\mathcal{E}$  which is Galois over  $E_1$  and  $E_3$ , then  $M$  is Galois over  $F$ .*

**Proof.** Fix an automorphism  $\sigma_3$  of the normal closure of  $M$  over  $F$  which restricts to the non-trivial automorphism of  $\mathcal{E}/E_3$ . Now if  $\sigma$  is any automorphism of the normal closure of  $M$  over  $F$ , the restriction of  $\sigma$  to  $\mathcal{E}$  must fix  $E_i$  for some  $i$ . If  $\sigma$  fixes  $E_1$  or  $E_3$ ,

then  $\sigma(M) = M$ , since  $M$  is Galois over  $E_1$  and  $E_3$ . Otherwise,  $\sigma$  must restrict to the non-trivial automorphism of  $\mathcal{E}/E_2$ . In this case,  $\sigma_3\sigma$  fixes  $E_1$ , so that  $\sigma_3\sigma(M) = M$ . Thus  $\sigma(M) = \sigma_3^{-1}(M) = M$ , and hence the only conjugate of  $M$  over  $F$  is itself, and  $M/F$  is Galois.  $\square$

For the next proposition, we denote the dihedral group of order 8 by  $D_8$ .

**Proposition 2.6.** *Let  $F$  be a totally real algebraic number field. Let  $E_1 = F(\sqrt{d_1})$  and  $E_3 = F(\sqrt{d_3})$  be distinct totally real quadratic extensions of  $F$  with composite  $\mathcal{E}$ , and assume that  $\mathcal{E} \neq E_1^{(1)}$ . Then there exists an element in  $K_2(\mathcal{E})$  which is simultaneously the image of elements of order 2 in  $K_2(E_1)$  and  $K_2(E_3)$ , but not the image of an element in  $K_2(F)$ , if and only if  $\mathcal{E}$  lies in a  $D_8$  Galois extension  $M$  of  $F$  which is biquadratic over  $E_1$ , biquadratic over  $E_3$ , and cyclic over  $E_2 = F(\sqrt{d_1d_3})$ . In particular, if such an element exists and is the image of  $\{-1, \alpha_3\}_{E_3}$ , then we can choose  $M = \mathcal{E}(\sqrt{\alpha_3})$ . Conversely, if such a field  $M = \mathcal{E}(\sqrt{\alpha})$  exists, then  $\{-1, \alpha\}_{\mathcal{E}}$  is an element satisfying the specified description.*

**Proof.** Let  $\pi_{E_1} \in E_1$  be the canonical element such that  $E_1(\sqrt{\pi_{E_1}}) = E_1^{(1)}$ . Under our assumption that  $E_1^{(1)} \neq \mathcal{E}$ , we have  $\mathcal{E}^{(1)} = \mathcal{E}(\sqrt{\pi_{E_1}})$ . Equivalently,

$$\pi_{\mathcal{E}} = \pi_{E_1}.$$

Suppose that there exist elements  $\{-1, \alpha_1\}_{E_1}$  and  $\{-1, \alpha_3\}_{E_3}$  with the same non-trivial image in  $K_2(\mathcal{E})$ . Then  $\alpha_1$  and  $\alpha_3$  are not in the Tate kernel of  $\mathcal{E}$ , but  $\alpha_1/\alpha_3$  is. We have seen that this Tate kernel is  $\langle \pi_{\mathcal{E}} \rangle \cdot (\mathcal{E}^\times)^2$ . So  $M = \mathcal{E}(\sqrt{\alpha_3})$  is a quadratic extension of  $\mathcal{E}$ , and is clearly biquadratic over  $E_3$  since  $\alpha_3 \in E_3$ . Also since  $\pi_{\mathcal{E}} = \pi_{E_1}$ , we see that  $\alpha_3 = \pi_{E_1}^t \gamma^2 \alpha_1$  for some integer  $t$  and  $\gamma \in \mathcal{E}^\times$ . Thus

$$M = \mathcal{E}(\sqrt{\alpha_3}) = \mathcal{E}\left(\sqrt{\pi_{E_1}^t \alpha_1}\right),$$

with  $\pi_{E_1}^t \alpha_1 \in E_1$ , so  $M$  is also biquadratic over  $E_1$ . Then by Lemma 2.5,  $M$  is Galois over  $F$ .

We now note that  $M$  is not abelian over  $F$ . If it were, then  $E_3(\sqrt{\alpha_3})/F$  would be Galois with group isomorphic to  $\text{Gal}(M/E_1)$ , which is the Klein four group. Thus we would have  $E_3(\sqrt{\alpha_3}) = E_3(\sqrt{a})$  for some  $a \in F$ . Consequently,  $\alpha_3 = a\eta^2$  for some  $\eta \in E_3^\times$ . This would imply that  $\{-1, \alpha_3\}_{E_3} = \{-1, a\}_{E_3}$ , the image of  $\{-1, a\}_F \in K_2(F)$ , so that  $\{-1, \alpha_3\}_{\mathcal{E}} = \{-1, a\}_{\mathcal{E}}$ . As we are assuming this is not the case, we must conclude that  $M/F$  is a Galois extension of degree 8 containing the non-Galois extension  $E_3(\sqrt{\alpha_3})/F$  of degree 4, and the only possible Galois group is  $D_8$ . This proves one implication of the proposition.

Conversely, suppose that an extension  $M = \mathcal{E}(\sqrt{\alpha})$  of the specified type exists. Since  $M$  is biquadratic over  $E_3$ , we must have

$$M = \mathcal{E}(\sqrt{\alpha_3}),$$

for some  $\alpha_3 \in E_3$ . By Kummer theory,  $\alpha_3$  must be  $\alpha$  times a square in  $\mathcal{E}$ , so that  $\{-1, \alpha\}_{\mathcal{E}} = \{-1, \alpha_3\}_{\mathcal{E}}$ . We show that the element  $\{-1, \alpha_3\}_{\mathcal{E}}$  satisfies the desired

conditions. First of all, it is the image of  $\{-1, \alpha_3\}_{E_3}$ . At the same time,  $M$  is also biquadratic over  $E_1$ , so

$$M = \mathcal{E}(\sqrt{\alpha_1}),$$

for some  $\alpha_1 \in E_1$  and, as above, we have  $\{-1, \alpha\}_{\mathcal{E}} = \{-1, \alpha_3\}_{\mathcal{E}}$ , which is clearly the image of  $\{-1, \alpha_1\}_{E_1} \in K_2(E_1)$ . It remains to show that  $\{-1, \alpha_3\}_{\mathcal{E}}$  is not the image of an element  $\omega \in K_2(F)$ . We proceed by contradiction. If such an  $\omega$  existed, then by Proposition 2.4 it would be of the form  $\omega = \{-1, a\}_F$ , for some  $a \in F$ . The condition that  $E_2 \neq F^{(1)}$  in that proposition is satisfied under our assumption that  $\mathcal{E} \neq E_1^{(1)}$ . Then we would have  $\{-1, a\}_{\mathcal{E}} = \{-1, \alpha_1\}_{\mathcal{E}}$ , so that  $\alpha_1/a$  is in the Tate kernel of  $\mathcal{E}$ . This yields that  $\alpha_1 = a\pi_{E_1}^t \eta^2$ , for some integer  $t$  and some  $\eta \in \mathcal{E}$ . Thus

$$M = \mathcal{E}(\sqrt{\alpha_1}) \subset \mathcal{E}(\sqrt{a}, \sqrt{\pi_{E_1}}).$$

But this field is abelian over  $F$ , while  $M$  is not, and we obtain the desired contradiction to complete the proof.  $\square$

**Corollary 2.7.** *Let  $F$  be a totally real algebraic number field. Let  $E_1 = F(\sqrt{d_1})$  and  $E_3 = F(\sqrt{d_3})$  be distinct totally real quadratic extensions of  $F$  with composite  $\mathcal{E}$ , and assume that  $\mathcal{E}$  does not contain  $F^{(1)}$ . Then there is no element of order 2 in  $K_2(\mathcal{E})$  which is simultaneously the image of elements of order 2 in each of  $K_2(E_1)$ ,  $K_2(E_2)$  and  $K_2(E_3)$ .*

**Proof.** The hypothesis on  $F^{(1)}$  implies that  $E_1^{(1)} \neq \mathcal{E} \neq E_2^{(1)}$ . If such an element existed, it would be of the form  $\{-1, \alpha_3\}_{\mathcal{E}}$ , with  $\alpha_3 \in E_3$ . By Proposition 2.6,  $\mathcal{E}(\sqrt{\alpha_3})$  would be both biquadratic and cyclic over  $E_2$ , a contradiction.  $\square$

Now let  $S_2$  denote the set of dyadic primes of  $F$ , i.e., the primes lying above 2. Note that the tame symbol  $(-1, \alpha)_{\mathfrak{p}}$  is trivial for all  $\alpha \in \mathcal{E}$  when  $\mathfrak{p}$  is dyadic, since  $-1$  is congruent to 1 modulo any dyadic prime. Thus  $K_2(\mathcal{O}_E^S) = K_2(\mathcal{O}_{\mathcal{E}}^{S \cup S_2})$ .

**Theorem 2.8 (Main Theorem for  $\mathcal{E}$  containing  $F^{(1)}$ ).** *Suppose that  $\mathcal{E}/F$  is a biquadratic extension of totally real number fields with the Birch-Tate conjecture holding for  $F$  and the three relative quadratic extensions of  $F$  in  $\mathcal{E}$ . Assume that one of these intermediate fields is  $E_1 = F^{(1)}$ . Let  $S$  contain the infinite primes of  $F$  and the primes that ramify in  $\mathcal{E}$ . Suppose that  $\mathcal{E}$  can be embedded in a  $D_8$  extension  $M$  of  $F$  which is biquadratic over  $E_1$  and unramified over  $F$  outside of  $S \cup S_2$ . Then  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  is contained in  $\text{Fit}_{\mathcal{E}/F}^S(1)$  with index 1 or 2.*

**Proof.** We let  $E_2$  and  $E_3$  be the other relative quadratic extensions of  $F$  in  $\mathcal{E}$  with  $M$  cyclic over  $E_2$  and biquadratic over  $E_3$ . Note that  $\mathcal{E} \neq E_1^{(1)}$ , for then  $\mathcal{E}$  would be cyclic over  $F$ . The conditions of Proposition 2.6 are met, so there exists an element  $\{-1, \alpha\}_{\mathcal{E}} \in K_2(\mathcal{E})$  which is the image of some  $\{-1, \alpha_1\}_{E_1} \in K_2(E_1)$  and of some  $\{-1, \alpha_3\}_{E_3} \in K_2(E_3)$ , but not in the image of  $K_2(F)$ . Since we have  $M = \mathcal{E}(\sqrt{\alpha})$  unramified over  $F$  outside of  $S \cup S_2$ ,  $\alpha$  has even valuation at all primes above those not in  $S \cup S_2$ . This implies that  $\{-1, \alpha\}_{\mathcal{E}}$  lies in the  $S$ -tame kernel  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  of  $\mathcal{E}$ . Since  $\mathcal{E}/E_i$  is unramified outside the primes above  $S$  for each  $i$ , we also find that  $\{-1, \alpha_1\}_{E_1} \in K_2(\mathcal{O}_{E_1}^S)$  and  $\{-1, \alpha_3\}_{E_3} \in K_2(\mathcal{O}_{E_3}^S)$ . Now Theorem 2.2 applies to give the result.  $\square$

**Theorem 2.9 (Main Theorem for  $\mathcal{E}$  not containing  $F^{(1)}$ ).** *Suppose that  $\mathcal{E}/F$  is a biquadratic extension of totally real number fields with the Birch-Tate conjecture holding for  $F$  and the relative quadratic extensions  $E_1, E_2$  and  $E_3$  of  $F$  in  $\mathcal{E}$ . Assume that  $F^{(1)}$  is not contained in  $\mathcal{E}$ . Let  $S$  contain the infinite primes of  $F$  and the primes that ramify in  $\mathcal{E}$ . Suppose that  $\mathcal{E}$  can be embedded in a  $D_8$  extension  $M$  of  $F$  which is cyclic over  $E_2$  and unramified over  $F$  outside of  $S \cup S_2$ , and also in a  $D_8$  extension  $M'$  of  $F$  which is cyclic over  $E_3$  and unramified over  $F$  outside of  $S \cup S_2$ . Then either  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  is contained in  $\text{Fit}_{\mathcal{E}/F}^S(1)$  with index 1 or 2, or  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  and  $\text{Fit}_{\mathcal{E}/F}^S(1)$  are both of index 2 in  $\text{Stick}_{\mathcal{E}/F}^S(-1) + \text{Fit}_{\mathcal{E}/F}^S(1)$ .*

**Proof.** Proposition 2.6 implies that there is an element  $\{-1, \alpha\}_{\mathcal{E}}$  in the intersection of the images in  $K_2(\mathcal{E})$  of  $K_2(E_1)$  and  $K_2(E_3)$  but not in the image of  $K_2(\mathcal{O}_F^S)$ . As in the proof of Theorem 2.8, the fact that  $M$  is unramified outside  $S$  implies that  $\{-1, \alpha\}_{\mathcal{E}}$  lies in the intersection of the images in  $K_2(\mathcal{O}_{\mathcal{E}}^S)$  of  $K_2(\mathcal{O}_{E_1}^S)$  and  $K_2(\mathcal{O}_{E_3}^S)$  but not in the image of  $K_2(\mathcal{O}_F^S)$ . The same argument with  $M$  replaced by  $M'$  then shows that we may apply Theorem 2.1 and obtain the desired conclusion.  $\square$

### 3. Applications

For easy reference, we first record some standard facts in a lemma.

**Lemma 3.1.** *Suppose that  $E/F$  is a relative quadratic extension and that  $\alpha$  and  $\beta$  lie in  $E^\times$ . Then*

- (a)  $E(\sqrt{\alpha}) = E(\sqrt{\beta})$  if and only if  $\alpha\beta$  is a square in  $E$ ;
- (b)  $E(\sqrt{\alpha})/F$  is a Galois extension if and only if the relative norm of  $\alpha$  is a square in  $E$ ;
- (c)  $E(\sqrt{\alpha})/F$  is a biquadratic extension if and only if  $\alpha$  is not a square in  $E$  and the relative norm of  $\alpha$  is a square in  $F$ .

**Proof.** First, (a) follows from Kummer theory or an easy exercise, while (b) follows from (a) upon taking  $\beta$  to be the conjugate of  $\alpha$  over  $F$ .

For (c), suppose that the extension is biquadratic. Then  $E(\sqrt{\alpha}) = E(\sqrt{a})$  for some  $a \in F$ . The implication follows upon applying (a) and taking the norm. For the converse, let  $c^2$  be the norm of  $\alpha$ . The automorphisms sending  $\sqrt{\alpha}$  to its conjugates  $\pm c/\sqrt{\alpha}$  both have order two, so cannot lie in a cyclic group.  $\square$

**Proposition 3.2.** *Let  $E_1$  be a totally real number field which is a relative quadratic extension of  $F$ . Let  $r$  be a totally positive non-square element of  $F$ , which is the norm of an integral element  $\alpha_1 \in E_1$  such that  $E_3 := F(\sqrt{r})$  is not contained in  $E_1^{(1)}$ . Set  $\mathcal{E} = E_1(\sqrt{r})$ , and let  $S$  contain all of the infinite primes of  $F$ , all of the primes that ramify in  $E_1$ , and all of the primes dividing  $r$ . Then  $M = \mathcal{E}(\sqrt{\alpha_1})$  is a  $D_8$  extension of  $F$  which is unramified outside of  $S \cup S_2$  and cyclic over  $E_2$ .*

**Proof.** The hypotheses clearly guarantee that  $E_3$  is a relative quadratic extension of  $F$ , distinct from  $E_1$ . Thus  $\mathcal{E}$  is a biquadratic extension of  $F$ , and we denote the

third relative quadratic extension of  $F$  in  $\mathcal{E}$  by  $E_2$ . Since  $F(\sqrt{r})$  is not contained in  $E_1^{(1)}$ , it is clear that  $\mathcal{E} \neq E_1^{(1)}$ , and since  $\alpha_1 \in E_1$ ,  $M$  is biquadratic over  $E_1$ . Because the relative norm of  $\alpha_1$  from  $\mathcal{E}$  to  $E_3$  is  $r$ , which is a square in  $E_3$ ,  $M$  is biquadratic over  $E_3$ , by Lemma 3.1. The relative norm of  $\alpha_1$  from  $\mathcal{E}$  to  $E_2$  is again  $r$ , which is a square in  $\mathcal{E}$ , but not a square in  $E_2$ . For this would imply that  $\sqrt{r} \in E_2$ , and consequently  $E_2 = F(\sqrt{r}) = E_3$ , a contradiction. Thus  $M$  is a cyclic extension of  $E_2$ . By Lemma 2.3, we conclude that  $M$  is a Galois extension of  $F$ . By Lemma 3.1 again, the extension  $E_1(\sqrt{\alpha_1})$  is not Galois over  $F$ , and one finds that  $M$  must be a  $D_8$  extension of  $F$ . Since  $\alpha_1$  is integral of norm  $r$ , the ramified primes of  $M = \mathcal{E}(\sqrt{\alpha_1})$  over  $\mathcal{E}$  are divisors of  $2r$ , and thus lie above primes in  $S \cup S_2$ .  $\square$

The following corollary strengthens and generalizes Corollary 5.3 of [9]. By Proposition 1.2, all of the assumptions of the Birch-Tate conjecture in both of these corollaries are satisfied when  $\mathcal{E}$  is absolutely abelian, for example if  $F = \mathbb{Q}$ .

**Corollary 3.3.** *Let  $F$  be a totally real field and let  $E_1 = F^{(1)}$ . Also let  $\alpha_1$  be an integral element of  $E_1$  whose norm to  $F$  is a totally positive non-square element  $r$  such that  $E_3 = F(\sqrt{r}) \neq E_1$ . Put  $\mathcal{E} = E_1 \cdot E_3$ , and let  $S$  contain all of the infinite primes of  $F$ , and all of the primes that ramify in  $\mathcal{E}/F$ . Assume that the Birch-Tate conjecture holds for  $F$  and that the quadratic extensions of  $F$  in  $E$ . Then we have*

$$\text{Stick}_{\mathcal{E}/F}^S(-1) \subset \text{Fit}_{\mathcal{E}/F}^S(1),$$

and the index is 1 or 2.

**Proof.** This follows from Proposition 3.2 and Theorem 2.8.  $\square$

**Corollary 3.4.** *Let  $F$  be a totally real field and  $d \in F$  be a totally positive integral element such that there is a unit  $\epsilon_1 \in E_1 = F(\sqrt{d})$  whose relative norm to  $F$  is  $-1$ . Fix an integral element  $\alpha_1 \in E_1$  whose relative norm  $r$  in  $F$  is totally positive, and not a square in  $E_1$ . Set  $E_3 = F(\sqrt{r})$  and  $\mathcal{E} = E_1 \cdot E_3$ , so  $E_2 = F(\sqrt{rd})$ , and suppose that  $F^{(1)}$  is not contained in  $\mathcal{E}$ . Let  $S$  contain all of the infinite primes of  $F$ , all of the primes that divide  $rd$ , and all of the dyadic primes that ramify in  $\mathcal{E}/F$ . Assume that the Birch-Tate conjecture holds for  $F$  and for the quadratic extensions of  $F$  in  $E$ . Then either  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  is contained in  $\text{Fit}_{\mathcal{E}/F}^S(1)$  with index 1 or 2, or  $\text{Stick}_{\mathcal{E}/F}^S(-1)$  and  $\text{Fit}_{\mathcal{E}/F}^S(1)$  are both of index 2 in  $\text{Stick}_{\mathcal{E}/F}^S(-1) + \text{Fit}_{\mathcal{E}/F}^S(1)$ .*

**Proof.** By Proposition 3.2,  $M = \mathcal{E}(\sqrt{\alpha_1})$  is a  $D_8$  extension of  $F$  which is unramified outside of  $S \cup S_2$  and cyclic over  $E_2$ . Now consider  $\alpha'_1 = \alpha_1 \epsilon_1 \sqrt{d}$ . The norm of  $\alpha'_1$  is  $r(-1)(-d) = rd$ , and  $E_2 = F(\sqrt{rd})$ . This time, Proposition 3.3 shows that  $M'$  is a  $D_8$  extension of  $F$  which is cyclic over  $E_3$  and unramified outside of  $S \cup S_2$ . The result follows from Theorem 2.9.  $\square$

## REFERENCES

- [1] P. Deligne and K. A. Ribet, *Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. **59** (1980), no. 3, 227–286.

- [2] H. Garland, *A finiteness theorem for  $K_2$  of a number field*, Ann. of Math. (2) **94** (1971), 534–548.
- [3] M. Kolster, *A relation between the 2-primary parts of the main conjecture and the Birch-Tate-conjecture*, Canad. Math. Bull. **32** (1989), no. 2, 248–251.
- [4] M. Kolster, *Cohomological version of the Lichtenbaum conjecture at the prime 2* (Appendix to *Two-primary algebraic K-theory of rings of integers in number fields*, by J. Rognes and C. Weibel), J. Amer. Math. Soc. **13** (2000), no. 1, 1–54.
- [5] M. Lescop, *Sur les 2-extensions de  $\mathbb{Q}$  dont la 2-partie du noyau sauvage est triviale*, Ph.D. thesis, University of Limoges, 2003.
- [6] B. Mazur and A. Wiles, *Class fields of abelian extensions of  $\mathbb{Q}$* , Invent. Math. **76** (1984), no. 2, 179–330.
- [7] D. Quillen, *Finite generation of the groups  $K_i$  of rings of algebraic integers*, In Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 179–198. Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973.
- [8] J. W. Sands, *L-values and the Fitting ideal of the tame kernel for relative quadratic extensions*, Acta Arith. **131** (2008), no. 4, 389–402.
- [9] J. W. Sands, *Values at  $s = -1$  of L-functions for multi-quadratic extensions of number fields, and the Fitting ideal of the tame kernel*, Int. J. Number Theory **5** (2009), no. 3, 383–405.
- [10] J. W. Sands and L. D. Simons, *Values at  $s = -1$  of L-functions for multi-quadratic extensions of number fields and annihilation of the tame kernel*, J. Lond. Math. Soc. (2) **76** (2007), no. 3, 545–555.
- [11] C. L. Siegel, *Über die Fourierschen Koeffizienten von Modulformen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1970** (1970), 15–56.
- [12] J. Tate, *Symbols in arithmetic*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 201–211, Gauthier-Villars, Paris, 1971.
- [13] J. Tate, *Relations between  $K_2$  and Galois cohomology*, Invent. Math. **36** (1976), 257–274.
- [14] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. (2) **131** (1990), no. 3, 493–540.

J. W. SANDS, DEPT. OF MATHEMATICS AND STATISTICS, 16 COLCHESTER AVE., U. OF VERMONT, BURLINGTON, VT 05401, USA.  
Jonathan.Sands@uvm.edu