VALUES AT $s = -1$ OF L-FUNCTIONS FOR MULTI-QUADRATIC EXTENSIONS OF NUMBER FIELDS, AND THE FITTING IDEAL OF THE TAME KERNEL

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Abstract. Fix a Galois extension $\mathcal{E}/F$ of totally real number fields such that the Galois group G has exponent 2. Let S be a finite set of primes of F containing the infinite primes and all those which ramify in $\mathcal{E}$, let $S_{\mathcal{E}}$ denote the primes of $\mathcal{E}$ lying above those in S, and let $\mathcal{O}_{\mathcal{E}}^S$ denote the ring of $S_{\mathcal{E}}$-integers of $\mathcal{E}$. We then compare the Fitting ideal of $K_2(\mathcal{O}_{\mathcal{E}}^S)$ as a $\mathbb{Z}[G]$-module with a higher Stickelberger ideal. The two extend to the same ideal in the maximal order of $\mathbb{Q}[G]$, and hence in $\mathbb{Z}[1/2][G]$. Results in $\mathbb{Z}[G]$ are obtained under the assumption of the Birch-Tate conjecture, especially for biquadratic extensions, where we compute the index of the higher Stickelberger ideal. We find a sufficient condition for the Fitting ideal to contain the higher Stickelberger ideal in the case where $\mathcal{E}$ is a biquadratic extension of F containing the first layer of the cyclotomic $\mathbb{Z}_2$-extension of F, and describe a class of biquadratic extensions of $F = \mathbb{Q}$ that satisfy this condition.

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I. Introduction

Fix an abelian Galois extension of number fields $\mathcal{E}/F$ and let G be the Galois group. Also fix a finite set S of primes of F which contains all of the infinite primes of F and all of the primes which ramify in $\mathcal{E}$. Associated with this data is a Stickelberger function, or equivariant L-function, $\theta^S_{\mathcal{E}/F}(s)$. It is a meromorphic function of s with values in the group ring $\mathbb{C}[G]$. To define it, let $\mathfrak{p}$ run through the (finite) primes of F not in S, and $\mathfrak{a}$ run through integral ideals of F which are relatively prime to each of the elements of S. Also let $\mathrm{N}\mathfrak{a}$ denote the absolute norm of the ideal $\mathfrak{a}$ and $\sigma_{\mathfrak{a}} \in G$ denote the well-defined automorphism attached to $\mathfrak{a}$ via the Artin map. Then

$$\theta^S_{\mathcal{E}/F}(s) = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a},S)=1}} \frac{1}{\mathrm{N}\mathfrak{a}^s}\sigma_{\mathfrak{a}}^{-1} = \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s}\sigma_{\mathfrak{p}}^{-1}\right)^{-1}.$$

These expressions converge for the real part of s greater than 1, and the function they define extends meromorphically to all of $\mathbb{C}$. When $\mathcal{E} = F$, the function $\theta^S_{F/F}(s)$ is simply the identity automorphism of F times $\zeta^S_F(s)$, the Dedekind zeta-function of F with Euler factors for the primes in S removed.

The function $\theta^S_{\mathcal{E}/F}(s)$ is connected with the arithmetic of the number fields $\mathcal{E}$ and F in ways one would like to make as precise as possible. The ring of S-integers $\mathcal{O}^S_F$ of F is defined to be the set of elements of F whose valuation is non-negative at every prime not in S. Similarly, define the ring $\mathcal{O}^S_{\mathcal{E}}$ of S-integers of $\mathcal{E}$ to be the set of elements of E whose valuation is non-negative at every prime not in $S_{\mathcal{E}}$, the set of all primes of $\mathcal{E}$ which lie above some prime in S. The function $\zeta_F^S(s)$ may be viewed as the zeta-function of the Dedekind domain $\mathcal{O}_F^S$.

We are interested in the "higher Stickelberger element" $\theta_{\mathcal{E}/F}^S(-1)$, which lies in $\mathbb{Q}[G]$ by the theorem of Klingen-Siegel [17], and is related to the algebraic K-group $K_2(\mathcal{O}_{\mathcal{E}}^S)$. This group is known to be finite by [4] and [13], and could be called the S-tame kernel of $\mathcal{E}$. It contains the tame kernel $K_2(\mathcal{O}_{\mathcal{E}})$ as a subgroup. Another piece of the arithmetic interpretation of $\theta_{\mathcal{E}/F}^S(-1)$ involves a group of roots of unity. Let μ_∞ denote the group of all roots of unity in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing $\mathcal{E}$, and let $\mathcal{G}$ denote the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. Define $W_2 = W_2(\overline{\mathbb{Q}})$ to be the $\mathbb{Z}[\mathcal{G}]$-module whose underlying group is μ_∞, with the action of $\gamma \in \mathcal{G}$ on $\omega \in W_2$ given by $\omega^\gamma = \gamma^2(\omega)$. For any subfield L of $\overline{\mathbb{Q}}$, let $W_2(L)$ be the submodule fixed under this action by the Galois group of $\overline{\mathbb{Q}}$ over L. Then $W_2(\mathcal{E})$ naturally becomes a $\mathbb{Z}[G]$-module, where the action of G arises by lifting elements of G to $\mathcal{G}$ and then using the action of $\mathcal{G}$ just defined. One easily sees that the G-fixed submodule $W_2(\mathcal{E})^G$ equals $W_2(F)$. We use the notation $w_2(L) = |W_2(L)|$, which we note is finite for any algebraic number field L.

Vital to our approach is the conjecture of Birch and Tate (see section 4 of [20]), which gives a precise arithmetic interpretation of $\zeta_F^S(-1)$. We state a form of it for an arbitrary finite set S which is easily seen to be equivalent to the original conjecture for the minimal choice of the set S, containing just the infinite primes (see Corollary 3.3 of [16]).

Conjecture 1.1 (Birch-Tate). *Suppose that F is totally real and the finite set S contains the infinite primes of F. Then*

$$\zeta_F^S(-1) = (-1)^{|S|}\frac{|K_2(\mathcal{O}_F^S)|}{w_2(F)}$$

Deep results on Iwasawa's Main conjecture in [11] and [22] lead to the following (see [8]).

Proposition 1.2. *The Birch-Tate Conjecture holds if F is abelian over $\mathbb{Q}$, and the odd part holds for all totally real F.*

Kolster [7] has shown that the 2-part of the Birch-Tate conjecture for F would follow from the 2-part of Iwasawa's Main conjecture for F.

For any module M over a ring R, we let $\mathrm{Ann}_R(M)$ denote the annihilator of M in R. The following result is proved in [15, Thm 1.3].

Proposition 1.3. *Let E/F be a relative quadratic extension of totally real number fields, with Galois group $\overline{G}$. Let S contain the infinite primes and those which ramify in E/F. Assume that the 2-part of the Birch-Tate conjecture holds for E and for F. Then the (first) Fitting ideal of $K_2(\mathcal{O}_E^S)$ as a $\mathbb{Z}[\overline{G}]$-module is*

$$\mathrm{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S)) = \mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1).$$

More specifically, this ideal equals its extension to the maximal order of $\mathbb{Q}[\overline{G}]$ if and only if it is not principal, and this happens exactly when E is not the first layer of the cyclotomic $\mathbb{Z}_2$-extension of F. Without the assumption of the Birch-Tate conjecture, the ideals $\mathrm{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S))$ and $\mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1)$ have the same extension to $\mathbb{Z}[1/2][\overline{G}]$.

In this paper, we build upon Proposition 1.3, obtaining more general results which suggest a close relationship between $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Fit}_{\mathbb{Z}[G]}(K_2(\mathcal{O}^S_{\mathcal{E}}))$ and the higher Stickelberger ideal $\mathrm{Stick}^S_{\mathcal{E}/F}(-1) = \mathrm{Ann}_{\mathbb{Z}[G]}(W_2(\mathcal{E}))\theta^S_{\mathcal{E}/F}(-1)$, particularly when G has exponent 2. Here the theorem of Deligne and Ribet [3] guarantees that $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ is an ideal in the integral group ring $\mathbb{Z}[G]$.

Part of the motivation for this work is to compare the situation concerning $K_2(\mathcal{O}^S_{\mathcal{E}})$ and $\theta^S_{\mathcal{E}/F}(-1)$, with that of the S-ideal class group of $\mathcal{E}$ and the Stickelberger element $\theta^S_{\mathcal{E}/F}(0)$. Non-triviality in the latter situation requires that $\mathcal{E}$ be a CM field and F be totally real. Then under the assumption of the Equivariant Tamagawa Number Conjecture, Greither [5] has recently obtained an equality of p-parts for odd primes p between an ideal built from Stickelberger elements, and the Fitting ideal of the Pontrjagin dual of the minus part of the ideal class group of $\mathcal{E}$. Thus one might wonder whether the Pontrjagin dual should occur in our investigation. So far, it does not. Taking the dual does not affect the Fitting ideal in the case of a cyclic group G, so the case of a non-cyclic group is of special interest to us. Our results tend to demonstrate the naturality of not taking the dual. For instance, the property of the Fitting ideal of the S-tame kernel with respect to change of extension field in Proposition 2.4 seems to have no straightforward counterpart if the S-tame kernel is replaced by its dual. Our main theorems provide relationships which do not involve the dual. However, these results are for groups of exponent 2. The situation may be different for groups whose p-part is non-cyclic for some odd prime p. We obtain fairly close relationships between $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ and $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$, but do not investigate the comparison with the Fitting ideal of the dual of $K_2(\mathcal{O}^S_{\mathcal{E}})$, leaving this to the future.

Of related interest is work of Barrett [1] comparing the image of $\mathrm{Fit}_{\mathbb{Z}[G]}(K_{2k}(\mathcal{O}_{\mathcal{E}}))$ in $\mathbb{Z}_p[G]$, for each odd prime p and positive integer k, with an ideal obtained using values of L-functions at the positive integer $k+1$ and a p-adic regulator. Assuming the Equivariant Tamagawa Number Conjecture and the Quillen-Lichtenbaum conjecture, he shows that the former ideal contains the latter. This result fits into the framework of Solomon's conjectures [19] for L-functions at $s = 1$.

II. Comparisons in Towers of Totally Real Fields

From now on, we assume that $\mathcal{E}$ is a totally real field. Let $\mathcal{R} = \mathbb{Z}[G]$. We will be considering abelian groups M whose operation is written multiplicatively, and which possess a natural G-action and therefore become $\mathcal{R}$-modules. For $\alpha \in \mathcal{R}$ and $m \in M$, we will write m^α for the action of α on m. If H is a subgroup of G, then $\mathbb{Z}[H]$ is a subring of $\mathcal{R}$, and we denote the augmentation ideal in $\mathbb{Z}[H]$ by I_H. Set $\overline{G} = G/H$. Note that $\mathcal{R}/(I_H)\mathcal{R} \cong \mathbb{Z}[\overline{G}]$.

Proposition 2.1 (Stickelberger ideals under change of extension field). *If E is any intermediate field between the totally real fields F and $\mathcal{E}$, and $\overline{G} = \mathrm{Gal}(E/F)$ then $\mathrm{Stick}^S_{E/F}(-1)$ equals the image of $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ under the natural projection map π from $\mathcal{R} = \mathbb{Z}[G]$ to $\overline{\mathcal{R}} = \mathbb{Z}[\overline{G}]$.*

Proof. It follows from [2, Lemma 2.3] that $\mathrm{Ann}_{\mathbb{Z}[G]}(W_2(\mathcal{E}))$ (respectively

$\mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E)))$ is generated by all elements of the form $\sigma_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2$ (respectively $\overline{\sigma}_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2 = \pi(\sigma_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2)$) for prime ideals $\mathfrak{q}$ not dividing $w_2(\mathcal{E})$ and the discriminant of $\mathcal{E}$. Hence $\mathrm{Stick}^S_{\mathcal{E}/F}(-1) = \mathrm{Ann}_{\mathbb{Z}[G]}(W_2(\mathcal{E}))\theta^S_{\mathcal{E}/F}(-1)$ (respectively $\mathrm{Stick}^S_{E/F}(-1) = \mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta^S_{E/F}(-1))$ is generated by all elements of the form $(\sigma_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2)\theta^S_{\mathcal{E}/F}(-1)$ (respectively $(\overline{\sigma}_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2)\theta^S_{E/F}(-1)$). Extend π by $\mathbb{Q}$-linearity; the inflation property of Artin L-functions implies that $\pi(\theta^S_{\mathcal{E}/F}(-1)) = \theta^S_{E/F}(-1)$. Hence $\mathrm{Stick}^S_{E/F}$ is generated by the elements $\pi(\sigma_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2)\pi(\theta^S_{\mathcal{E}/F}(-1)) = \pi\big((\sigma_{\mathfrak{q}} - \mathrm{N}\,\mathfrak{q}^2)(\theta^S_{\mathcal{E}/F}(-1))\big)$, and the result follows, since π is surjective.

Remark 2.2. The proof of Proposition 2.1 easily generalizes to higher Stickelberger ideals for any non-positive integer $-k$. One replaces $\theta^S_{\mathcal{E}/F}(-1)$ by $\theta^S_{\mathcal{E}/F}(-k)$ and $W_2(\mathcal{E})$ by $W_{k+1}(\mathcal{E})$, defined as those roots of unity fixed by the $k+1$st powers of all automorphisms of $\overline{\mathbb{Q}}$ over $\mathcal{E}$. The Deligne-Ribet theorem is valid here as well to guarantee that this yields an ideal $\mathrm{Stick}^S_{\mathcal{E}/F}(-k)$ of $\mathbb{Z}[G]$. This ideal is expected to be related to $\mathrm{Fit}^S_{\mathcal{E}/F}(k) = \mathrm{Fit}_{\mathbb{Z}[G]}(K_{2k}(\mathcal{O}^S_{\mathcal{E}}))$ (see [18]), although a closer relationship is expected upon replacing $K_{2k}(\mathcal{O}^S_{\mathcal{E}})$ by an appropriate étale cohomology group. In our case of $k = 1$, the two are known to be the same.

Proposition 2.3. *Suppose that $\mathcal{E}/E$ is an abelian Galois extension of totally real number fields with group H. Suppose also that S contains all of the infinite primes of E and the primes which ramify in $\mathcal{E}/E$. Let I_H denote the augmentation ideal of $\mathbb{Z}[H]$. Then the transfer map from $K_2(\mathcal{O}^S_{\mathcal{E}})$ to $K_2(\mathcal{O}^S_E)$ induces an isomorphism of $\mathbb{Z}[\overline{G}]$-modules*

$$K_2(\mathcal{O}^S_{\mathcal{E}})/K_2(\mathcal{O}^S_{\mathcal{E}})^{I_H} \cong K_2(\mathcal{O}^S_E).$$

Proof. Under our assumptions on S, it follows more generally from Kahn's theorem of [6, 5.1] as observed in [9] that the transfer map from $K_2(\mathcal{O}_{\mathcal{E}}^S)$ to $K_2(\mathcal{O}_E^S)$ has kernel $K_2(\mathcal{O}_{\mathcal{E}}^S)^{I_H}$ and a cokernel which is an elementary abelian 2-group of rank equal to the number of real primes of E which ramify in $\mathcal{E}$. Here that rank is zero, and the result is established.

Proposition 2.4 (Fitting ideals of S-tame kernels under change of extension field). *Suppose that $\mathcal{E}/F$ is an abelian Galois extension of totally real number fields with group G, and E is an intermediate field with* $\mathrm{Gal}(E/F)$ *denoted by $\overline{G}$. Suppose also that S contains all of the infinite primes of F and the primes which ramify in $\mathcal{E}$. Then* $\mathrm{Fit}^S_{E/F}(1)$ *equals the image of* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *under the natural projection map π from $\mathcal{R} = \mathbb{Z}[G]$ to $\overline{\mathcal{R}} = \mathbb{Z}[\overline{G}]$.*

Proof. From Proposition 2.3, we get

$$\mathrm{Fit}^S_{E/F}(1) = \mathrm{Fit}_{\overline{\mathcal{R}}}(K_2(\mathcal{O}_E^S)) = \mathrm{Fit}_{\overline{\mathcal{R}}}(K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{I_H}).$$

Identifying $\overline{\mathcal{R}}$ with $\mathcal{R}/(I_H)\mathcal{R}$, this becomes

$$\mathrm{Fit}_{\mathcal{R}/(I_H)\mathcal{R}}(K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{I_H}) = \mathrm{Fit}_{\mathcal{R}/(I_H)\mathcal{R}}(K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{\mathcal{R}(I_H)}).$$

Now a standard property of Fitting ideals implies that this ideal equals the image of $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Fit}_{\mathcal{R}}(K_2(\mathcal{O}_{\mathcal{E}}^S))$ in $\mathcal{R}/(I_H)\mathcal{R}$, and under our identifications, this corresponds to $\pi(\mathrm{Fit}^S_{\mathcal{E}/F}(1))$ in $\overline{\mathcal{R}}$, as desired.

Theorem 2.5 (First Comparison Theorem). *Suppose that $\mathcal{E}/F$ is an abelian Galois extension of totally real number fields with group G, and E is a relative quadratic extension of F in $\mathcal{E}$. If the Birch-Tate conjecture* *holds for E and F, then* $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ *and* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *in* $\mathbb{Z}[G]$ *project to the same image in* $\mathbb{Z}[\mathrm{Gal}(E/F)]$*. Equality of the images in* $\mathbb{Z}[1/2][G]$ *holds unconditionally.*

Proof. Let $\overline{G} = \mathrm{Gal}(E/F)$ and apply Propositions 2.1 and 2.3. The images of $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ in $\mathbb{Z}[\overline{G}]$ are $\mathrm{Stick}^S_{E/F}(-1)$ and $\mathrm{Fit}^S_{E/F}(1)$, respectively. Proposition 1.3 completes the proof.

Proposition 2.6 (Stickelberger ideals under change of base field). *Suppose that $\mathcal{F}$ is an intermediate field between F and $\mathcal{E}$. Then*

$$\mathrm{Stick}^S_{\mathcal{E}/\mathcal{F}}(-1) \subset \mathrm{Stick}^S_{\mathcal{E}/F}(-1).$$

Proof. This is a special case of the Corollary to the Main Theorem of [14].

Proposition 2.7 (Fitting ideals under change of base field). *Suppose that $\mathcal{F}$ is an intermediate field between the totally real fields F and $\mathcal{E}$. Then*

$$\mathrm{Fit}^S_{\mathcal{E}/\mathcal{F}}(1) \subset \mathrm{Fit}^S_{\mathcal{E}/F}(1).$$

Proof. This is a direct application of a property of Fitting ideals for subrings which follows immediately from the definition.

Theorem 2.8 (Second Comparison Theorem). *Suppose that $\mathcal{F}$ is an intermediate field between F and $\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is of degree 2. If the Birch-Tate conjecture holds for $\mathcal{E}$ and $\mathcal{F}$, then* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *and* $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ *both contain* $\mathrm{Fit}^S_{\mathcal{E}/\mathcal{F}}(1) = \mathrm{Stick}^S_{\mathcal{E}/\mathcal{F}}(-1)$*. Without the assumption of the Birch-Tate conjecture, the extensions of* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *and* $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ *to* $\mathbb{Z}[1/2][G]$ *both contain* $\mathrm{Stick}^S_{\mathcal{E}/\mathcal{F}}(-1)$.

Proof. Assuming the Birch-Tate conjecture, the equality $\mathrm{Fit}^S_{\mathcal{E}/\mathcal{F}}(1) = \mathrm{Stick}^S_{\mathcal{E}/\mathcal{F}}(-1)$ holds by Proposition 1.3. Apply Propositions 2.6 and 2.7 to obtain the result. Without assuming the Birch-Tate conjecture, the proof goes through after extending ideals to the group rings obtained by adjoining 1/2.

III. Comparisons in the Maximal Order $\mathcal{S}$ of $\mathbb{Q}[G]$ when $G^2 = 1$

From now on we assume that the abelian group G has exponent 2, and order 2^m. Then $\mathcal{E}$ is a composite of relative quadratic extensions of F, and $\mathcal{E}/F$ is what we call a multi-quadratic extension. In this case, a non-trivial element χ in the character group $\hat{G}$ of G must have order 2, and a kernel $\ker(\chi)$ of index 2 in G. We denote the fixed field of $\ker(\chi)$ by E_χ. It is a relative quadratic extension of F. For each non-trivial χ, we fix an element $\tau_\chi \in G$ which does not lie in $\ker(\chi)$. Then τ_χ restricts to the non-trivial automorphism of E_χ/F. Denote the trivial character of G by χ_0.

Each χ is associated with an idempotent $e_\chi = \frac{1}{2^m}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1} \in \mathbb{Q}[G]$. The maximal order of $\mathcal{R} = \mathbb{Z}[G]$ in $\mathbb{Q}[G]$ is $\mathcal{S} = \oplus_\chi \mathbb{Z}e_\chi$. Let I_χ denote the kernel of the natural map from $\mathcal{R}$ to $\mathcal{R}e_\chi = \mathbb{Z}e_\chi$. It is easy to see that I_χ is generated by the elements $\sigma - \chi(\sigma)$ as σ ranges over G. Note that as $\mathcal{R}$-algebras, we have

$$\mathcal{S} \cong \bigoplus\nolimits_{\chi\in\hat{G}} \mathcal{R}e_\chi \cong \bigoplus\nolimits_{\chi} \mathcal{R}/I_\chi\mathcal{R}.$$

To simplify notation, we now set $k_2^S(E) = |K_2(\mathcal{O}_E^S)|$. For any $\mathcal{R}$-module M and $\alpha \in \mathcal{R}$, let M_α denote the submodule of elements annihilated by α. For the relative quadratic extensions E_χ/F, we also set $k_2^S(E_\chi)^- = |K_2(\mathcal{O}^S_{E_\chi})_{1+\tau_\chi}|$ and $w_2(E_\chi)^- = |W_2(E_\chi)_{1+\tau_\chi}|$.

Proposition 3.1 (The Stickelberger element in a multi-quadratic extension). *Assuming that the Birch-Tate conjecture holds for each E_χ and for F, we have*

$$(-1)^{|S|}\theta^S_{\mathcal{E}/F}(-1) = \frac{k_2^S(F)}{w_2(F)} e_{\chi_0} + \sum_{\chi \neq \chi_0} (-1)^{|S_{E_\chi}|} \frac{w_2(F)}{w_2(E_\chi)} \frac{k_2^S(E_\chi)}{k_2^S(F)} e_\chi.$$

Equality up to factors of 2 in each component holds unconditionally.

Proof. This follows from standard properties of L-functions and an application of Proposition 1.2. See [16, Proposition 5.1] for more details.

Lemma 3.2. *Suppose that M is a finite $\mathcal{R}$-module. Then* $\mathrm{Fit}_{\mathcal{R}}(M)\mathcal{S} = \bigoplus_{\chi \in \hat{G}} |M/I_\chi M| \mathbb{Z} e_\chi$.

Proof. Note that $\mathcal{R}e_\chi = \mathbb{Z}e_\chi \cong \mathbb{Z}$. Then using standard properties of Fitting ideals and tensor products, we have

$$\begin{aligned}\mathrm{Fit}_{\mathcal{R}}(M)\mathcal{S} &= \mathrm{Fit}_{\mathcal{S}}(M \otimes_{\mathcal{R}} \mathcal{S}) = \mathrm{Fit}_{\mathcal{S}}(M \otimes_{\mathcal{R}} (\bigoplus\nolimits_\chi \mathcal{R}/I_\chi)) \\ &= \mathrm{Fit}_{\oplus_\chi \mathcal{R}e_\chi}(\bigoplus\nolimits_\chi M/I_\chi M) = \bigoplus\nolimits_\chi \mathrm{Fit}_{\mathbb{Z}e_\chi}(M/I_\chi M) \\ &= \bigoplus\nolimits_\chi |M/I_\chi M| \mathbb{Z} e_\chi.\end{aligned}$$

Lemma 3.3. *Suppose that E is an extension of F contained in $\mathcal{E}$ such that $\mathcal{E}/E$ is a quadratic extension. Let τ be a generator of* $\mathrm{Gal}(\mathcal{E}/E)$*. Then as $\mathbb{Z}[G]$-modules, $W_2(\mathcal{E})/W_2(\mathcal{E})^{1-\tau} \cong W_2(E)$ and $W_2(\mathcal{E})/W_2(\mathcal{E})^{1+\tau} \cong W_2(\mathcal{E})_{1+\tau}$. Also* $|W_2(\mathcal{E})_{1+\tau}| \equiv 2 \pmod 4$.

Proof. From the exact sequence

$$1 \to W_2(E) \to W_2(\mathcal{E}) \stackrel{1-\tau}{\to} W_2(\mathcal{E})^{1-\tau} \to 1,$$

we see that the first pair of modules to be shown isomorphic indeed have the same order. Since $W_2(\mathcal{E})$ is a cyclic group, the isomorphism in question is now clearly induced by raising to the power $|W_2(\mathcal{E})^{1-\tau}|$. The proof of the second isomorphism is similar.

For the last statement, clearly $\sqrt{-1} \in W_2(\mathbb{Q})$ and hence τ acts trivially on $\sqrt{-1}$. Thus $\sqrt{-1}^{1+\tau} = \sqrt{-1}^2 = -1 \neq 1$. This shows that $\sqrt{-1} \notin W_2(\mathcal{E})_{1+\tau}$, so $4 \nmid |W_2(\mathcal{E})_{1+\tau}|$. However, $|W_2(\mathcal{E})_{1+\tau}|$ must be even, as $-1 \in W_2(\mathcal{E})_{1+\tau}$.

Corollary 3.4. *$W_2(\mathcal{E})/W_2(\mathcal{E})^{I_\chi}$ is isomorphic as a $\mathbb{Z}[G]$-module to $W_2(F)$ when $\chi = \chi_0$ and to $W_2(E_\chi)_{1+\tau_\chi}$ otherwise.*

Proof. The ideal I_χ is generated by the elements $1 - \sigma_i$ for σ_i running through a minimal set of generators of $\ker(\chi)$, together with, in the case of $\chi \neq \chi_0$, one element $1 + \sigma_m$ for $\sigma_m = \tau_\chi \notin \ker(\chi)$. Thus the result follows from successive application of Lemma 3.3 with $1 - \tau = 1 - \sigma_i$ for $i = 1, \ldots, m-1$ followed by one more application with $1 \pm \tau = 1 \pm \sigma_m$.

Proposition 3.5 (Extension of $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ to $\mathcal{S}$).

$$\mathrm{Ann}_{\mathcal{R}} W_2(\mathcal{E})\mathcal{S} = w_2(F)\mathbb{Z}e_{\chi_0} \oplus \bigoplus_{\chi \neq \chi_0} w_2(E_\chi)^- \mathbb{Z}e_\chi.$$

Proof. Since $W_2(\mathcal{E})$ is a cyclic group, and hence a cyclic $\mathcal{R}$-module,

$$\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E})) = \mathrm{Fit}_{\mathcal{R}}(W_2(\mathcal{E})).$$

By Lemma 3.2,

$$\mathrm{Fit}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S} = \bigoplus_{\chi\in\hat{G}} |W_2(\mathcal{E})/W_2(\mathcal{E})^{I_\chi}|\mathbb{Z}e_\chi.$$

Applying Corollary 3.4, we find that

$$\mathrm{Ann}_{\mathcal{R}} W_2(\mathcal{E})\mathcal{S} = w_2(F)\mathbb{Z}e_{\chi_0} \oplus \bigoplus_{\chi\neq\chi_0} |W_2(E_\chi)_{1+\tau_\chi}|\mathbb{Z}e_\chi.$$

For each positive integer j, let $F^{(j)}$ denote the jth layer of the cyclotomic $\mathbb{Z}_2$-extension of F. It is cyclic of degree 2^j over F. Since $G = \mathrm{Gal}(\mathcal{E}/F)$ has exponent 2, $\mathcal{E}$ cannot contain $F^{(2)}$. For relative quadratic extensions E/F in $\mathcal{E}$, the case where $E = F^{(1)}$ (if this lies in $\mathcal{E}$), tends to be rather special. For example, we will make repeated use of the following result.

Proposition 3.6 (Cohomology of $W_2(E)$ and $K_2(\mathcal{O}_E^S)$ for a relative quadratic extension). *Suppose that E/F is a relative quadratic extension of totally real fields, and τ is the non-trivial automorphism of E over F. Let the finite set S of primes of F contain the infinite and ramified primes as usual. Let $\iota_{E/F} : K_2(\mathcal{O}_F^S) \to K_2(\mathcal{O}_E^S)_{1-\tau}$ be the natural map induced by the inclusion of rings, with kernel $\ker(\iota_{E/F})$ and cokernel $\mathrm{coker}(\iota_{E/F})$. Then*

$$\begin{aligned} |W_2(E)_{1-\tau}/W_2(E)^{1+\tau}| &= |W_2(E)_{1+\tau}/W_2(E)^{1-\tau}| \\ &= |K_2(\mathcal{O}_E^S)_{1-\tau}/K_2(\mathcal{O}_E^S)^{1+\tau}| = |K_2(\mathcal{O}_E^S)_{1+\tau}/K_2(\mathcal{O}_E^S)^{1-\tau}| \\ &= |\ker(\iota_{E/F})| = |\mathrm{coker}(\iota_{E/F})|. \end{aligned}$$

If $E = F^{(1)}$, then the common value of these quantities is 1; otherwise it is 2. In either case, $W_2(E)_{1-\tau} = W_2(F)$ and $|K_2(\mathcal{O}_E^S)_{1-\tau}| = |K_2(\mathcal{O}_F^S)|$.

Proof. See [15, Propositions 3.4, 2.2 and 2.3].

Lemma 3.7. *Assuming the Birch-Tate conjecture for E_χ and F, we have*

$$w_2(F)e_{\chi_0}\theta^S_{\mathcal{E}/F}(-1) = \pm k_2^S(F)e_{\chi_0}$$

and for $\chi \neq \chi_0$,

$$w_2(E_\chi)^- e_\chi \theta^S_{\mathcal{E}/F}(-1) = \pm k_2^S(E_\chi)^- e_\chi.$$

Equality of odd parts holds unconditionally.

Proof. (Adapted from [15, Lemma 7.2]) Using Proposition 3.1, we see that

$$\pm w_2(F)e_{\chi_0}\theta^S_{\mathcal{E}/F}(-1) = w_2(F)\frac{k_2^S(F)}{w_2(F)}e_{\chi_0} = k_2^S(F)e_{\chi_0}.$$

Similarly, and using Lemma 3.3 as well, we have

$$\begin{aligned} &\pm w_2(E_\chi)^- e_\chi \theta^S_{\mathcal{E}/F}(-1) \\ &= |W_2(E_\chi)_{1+\tau_\chi}| \frac{w_2(F)}{w_2(E_\chi)} \frac{|K_2(\mathcal{O}^S_{E_\chi})|}{|K_2(\mathcal{O}^S_F)|} e_\chi = \frac{|W_2(E_\chi)_{1-\tau_\chi}|}{|W_2(E_\chi)^{1+\tau_\chi}|} \frac{|K_2(\mathcal{O}^S_{E_\chi})|}{|K_2(\mathcal{O}^S_F)|} e_\chi. \end{aligned}$$

At this point, we apply Proposition 3.6 to E_χ/F and obtain that $|K_2(\mathcal{O}^S_F)| = |K_2(\mathcal{O}^S_{E_\chi})_{1-\tau_\chi}|$, while the cohomology groups of $K_2(\mathcal{O}^S_{E_\chi})$ and $W_2(E\chi)$ over $\mathrm{Gal}(E_\chi/F)$ have the same order. The last expression now becomes

$$\frac{|K_2(\mathcal{O}^S_{E_\chi})_{1-\tau_\chi}|}{|K_2(\mathcal{O}^S_{E_\chi})^{1+\tau_\chi}|} \frac{|K_2(\mathcal{O}^S_{E_\chi})|}{|K_2(\mathcal{O}^S_{E_\chi})_{1-\tau_\chi}|} e_\chi = \frac{|K_2(\mathcal{O}^S_{E_\chi})|}{|K_2(\mathcal{O}^S_{E_\chi})^{1+\tau_\chi}|} e_\chi = |K_2(\mathcal{O}^S_{E_\chi})_{1+\tau_\chi}| e_\chi$$

All equalities are valid up to factors of two without the assumption of the Birch-Tate conjecture.

Proposition 3.8 (Extension of the Stickelberger ideal to $\mathcal{S}$). *Assuming the Birch-Tate conjecture for each E_χ and for F, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$ is generated as an $\mathcal{S}$-ideal by*

$$k_2^S(F)e_{\chi_0} + \sum_{\chi\neq\chi_0} k_2^S(E_\chi)^- e_\chi.$$

Without assuming the Birch-Tate conjecture, this element is a generator for $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}[1/2]$ in $\mathcal{S}[1/2] = \mathcal{R}[1/2]$.

Proof. Using Proposition 3.5,

$$\begin{aligned}\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S} &= \theta^S_{\mathcal{E}/F}(-1)\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S} \\ &= \theta^S_{\mathcal{E}/F}(-1)\big(w_2(F)e_{\chi_0} + \sum_{\chi\neq\chi_0} w_2(E_\chi)^- e_\chi\big)\mathcal{S}\end{aligned}$$

Since $\mathcal{S} = \oplus\chi\mathbb{Z}e_\chi$, it suffices to consider each component of $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$ individually. Proposition 3.7 completes the proof under the assumption of the conjecture. For the unconditional part, note that the same proof applies in $\mathcal{S}[1/2] = \sum_\chi \mathbb{Z}[1/2]e_\chi$.

Proposition 3.9 (Extension of the Fitting ideal to $\mathcal{S}$).

$$\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathbb{Z}k_2^S(F)e_{\chi_0} \oplus \bigoplus\nolimits_{\chi\neq\chi_0} \mathbb{Z}k_2^S(E_\chi)^- e_\chi$$

Proof. Lemma 3.2 gives

$$\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \bigoplus\nolimits_{\chi\in\hat{G}} |K_2(\mathcal{O}^S_{\mathcal{E}})/K_2(\mathcal{O}^S_{\mathcal{E}})^{I_\chi}|\mathbb{Z}e_\chi.$$

Since $I_{\chi_0} = I_G$, while I_χ is generated by $I_{\ker(\chi)}$ and $1+\tau_\chi$ for $\chi\neq\chi_0$, Proposition 2.3 allows us to deduce that

$$K_2(\mathcal{O}^S_{\mathcal{E}})/K_2(\mathcal{O}^S_{\mathcal{E}})^{I_{\chi_0}} \cong K_2(\mathcal{O}^S_F)$$

and

$$K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{I_\chi} \cong K_2(\mathcal{O}_{E_\chi}^S)/K_2(\mathcal{O}_{E_\chi}^S)^{1+\tau_\chi}.$$

Then consideration of the surjective mapping from $K_2(\mathcal{O}_{E_\chi}^S)$ to its image under $1+\tau_\chi$ shows that $|K_2(\mathcal{O}_{E_\chi}^S)|/|K_2(\mathcal{O}_{E_\chi}^S)^{1+\tau_\chi}| = |K_2(\mathcal{O}_{E_\chi})_{1+\tau_\chi}^S|$. This establishes the result.

Theorem 3.10. (Third Comparison Theorem) $\mathrm{Fit}_{\mathcal{E}/F}^S(1)$ *and* $\mathrm{Stick}_{\mathcal{E}/F}^S(-1)$ *extend to the same ideal in* $\mathbb{Z}[1/2][G]$*. Assuming the Birch-Tate conjecture for* F *and for each* E_χ*, they extend to the same ideal in* $\mathcal{S}$*.*

Proof. The second statement follows directly from Proposition 3.8 and Proposition 3.9. The first follows from these two by taking images in $\mathcal{S}[1/2] = \mathcal{R}[1/2] = \mathbb{Z}[1/2][G]$.

Lemma 3.11. *Let* p^t *be a positive power of a prime, and* E *be a totally real number field. Denote a primitive* p^t*th root of unity by* ω_{p^t}*. For* p *odd, we have* $p^t | w_2(E) \iff \omega_{p^t} + \omega_{p^t}^{-1} \in E$*. For* $p = 2$*, we have* $2^t | w_2(E) \iff \omega_{2^{t-1}} + \omega_{2^{t-1}}^{-1} \in E$*.*

Proof. If p is odd and $p^t | w_2(E)$, then by the definition of $w_2(E)$, $\mathrm{Gal}(E(\omega_{p^t})/E)$ has exponent 2. However, this group contains complex conjugation and injects into the cyclic group $(\mathbb{Z}/p^t\mathbb{Z})^\times$, so is generated by complex conjugation. Thus E contains the trace $\omega_{p^t} + \omega_{p^t}^{-1}$. Conversely, if E contains $\omega_{p^t} + \omega_{p^t}^{-1}$, then $E(\omega_{p^t})/E$ has degree 2, and $p^t | w_2(E)$. The proof for $p = 2$ requires just a slight modification (see [16, Lemma 7.2]).

Corollary 3.12. *Let* p^t *be a positive power of a prime.*

1. *If* $p^t|w_2(\mathcal{E})$*, then* $p^t|w_2(E_\chi)$ *for some non-principal* χ*.*

2. *If* $p^t|w_2(\mathcal{E})$ *and* p^{t-1} *exactly divides* $w_2(F)$*, then* $p|w_2(E_\chi)$ *for a unique non-principal* χ*.*

Proof. We assume p is odd. For the proof when $p = 2$, simply replace t by $t-1$ as appropriate, according to Lemma 3.11.

1. Assume that $p^t|w_2(\mathcal{E})$. By Lemma 3.11, $\omega_{p^t} + \omega_{p^t}^{-1} \in \mathcal{E}$. Thus $F(\omega_{p^t} + \omega_{p^t}^{-1}) \subset \mathcal{E}$, which is multi-quadratic over F. Hence $\mathrm{Gal}(F(\omega_{p^t} + \omega_{p^t}^{-1})/F)$ has exponent 2. However this group also injects into the cyclic group $(\mathbb{Z}/p^t\mathbb{Z})^\times/\pm 1$, so must have order 1 or 2. So $F(\omega_{p^t} + \omega_{p^t}^{-1})$ is contained in a quadratic extension of F inside $\mathcal{E}$, and hence lies in one of the E_χ. By Lemma 3.11 again, $p^t|w_2(E_\chi)$.

2. Let p^{t-1} exactly divide $w_2(F)$. Then $p^t|w_2(\mathcal{E})$ and so by part (1), $p^t|w_2(E_\chi)$ for some χ. By Lemma 3.11, we now have that F does not contain $\omega_{p^t} + \omega_{p^t}^{-1} \in E_\chi$. Since E_χ/F is relative quadratic, we must have $E_\chi = F(\omega_{p^t} + \omega_{p^t}^{-1})$. However, this specifies E_χ, and hence χ, uniquely. Lemma 3.11 then implies that $p^t|w_2(E_\chi)$ for this χ and no other. The result follows.

Lemma 3.13. *Assume the Birch-Tate conjecture holds for* F*, each* $E\chi$*, and* $\mathcal{E}$*. Then*

$$\frac{|K_2(\mathcal{O}_{\mathcal{E}}^S)|/k_2^S(F)}{w_2(\mathcal{E})/w_2(F)} = \prod_{\chi\neq\chi_0} \frac{|K_2(\mathcal{O}_{E_\chi}^S)|/k_2^S(F)}{w_2(E_\chi)/w_2(F)}.$$

Without the assumption of the Birch-Tate conjecture, equality holds for the odd parts.

Proof. Standard properties of Artin L-functions give

$$\zeta_{\mathcal{E}}^S(s) = \zeta_F^S(s) \prod_{\chi \neq \chi_0} L_{\mathcal{E}/F}^S(s, \chi) = \zeta_F^S(s) \prod_{\chi \neq \chi_0} (\zeta_{E_\chi}^S(s)/\zeta_F^S(s)).$$

Setting $s = -1$ and applying the Birch-Tate conjecture or just its proven odd part gives the result.

Proposition 3.14 (The index of the higher Stickelberger ideal in a multi-quadratic extension). *Assume the Birch-Tate conjecture holds for F, each $E\chi$, and $\mathcal{E}$. Let $\delta = \delta_{\mathcal{E}/F} = 2$ if $F^{(1)} \subset \mathcal{E}$, and $\delta_{\mathcal{E}/F} = 1$ otherwise. Then*

$$(\mathcal{R} : \mathrm{Stick}_{\mathcal{E}/F}^S(-1)) = |K_2(\mathcal{O}_{\mathcal{E}}^S)| \frac{(\mathrm{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S} : \mathrm{Stick}_{\mathcal{E}/F}^S(-1))}{\delta_{\mathcal{E}/F} 2^{(m-2)2^{m-1}+1}}.$$

Without the assumption of the Birch-Tate conjecture, one has that

$$(\mathcal{R} : \mathrm{Stick}_{\mathcal{E}/F}^S(-1)) = 2^c |K_2(\mathcal{O}_{\mathcal{E}}^S)|$$

for some integer c.

Proof. By Proposition 3.8, we have

$$(\mathcal{S} : \mathrm{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S}) = k_2^S(F) \prod_{\chi \neq \chi_0} k_2^S(E_\chi)^-.$$

Applying Proposition 3.6,

$$\begin{aligned} k_2^S(E_\chi)^- = |K_2(\mathcal{O}_{E_\chi}^S)_{1+\tau_\chi}| = |K_2(\mathcal{O}_{E_\chi}^S)|/|K_2(\mathcal{O}_{E_\chi}^S)^{1+\tau_\chi}| \\ = \delta_\chi |K_2(\mathcal{O}_{E_\chi}^S)|/k_2^S(F), \end{aligned}$$

where $\delta_\chi = 1$ if $\mathcal{E}_\chi = F^{(1)}$, and $\delta_\chi = 2$ otherwise. Hence

$$(\mathcal{S} : \mathrm{Stick}_{\mathcal{E}/F}^S(-1)\mathcal{S}) = (2^{2^m-1}/\delta) k_2^S(F) \prod_{\chi \neq \chi_0} |K_2(\mathcal{O}_{E_\chi}^S)|/k_2^S(F).$$

Using Lemma 3.13 then gives us

$$(\mathcal{S} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}) = (2^{2^m-1}/\delta)|K_2(\mathcal{O}^S_E)|\frac{\prod_{\chi\neq\chi_0} w_2(E_\chi)/w_2(F)}{w_2(\mathcal{E})/w_2(F)}.$$

Here, we claim that the term $\frac{\prod_{\chi\neq\chi_0} w_2(E_\chi)/w_2(F)}{w_2(\mathcal{E})/w_2(F)}$ equals 1. For, if p^t divides the denominator, then p^t divides the numerator, by Proposition 3.12(1). On the other hand, if p^t divides the numerator, then it divides only one term in the numerator, by Proposition 3.12(2). Since $W_2(E_\chi) \subset W_2(\mathcal{E})$, it is then clear that p^t divides the denominator. We conclude that

$$(\mathcal{S} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}) = (2^{2^m-1}/\delta)|K_2(\mathcal{O}^S_E)|.$$

A determinant calculation using the orthogonality relations for characters shows that

$$(\mathcal{S} : \mathcal{R}) = 2^{m2^{m-1}}.$$

Combining these clearly yields the result. Without the assumption of the Birch-Tate conjecture, all equalities hold up to powers of 2, and $(\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1))$ is also a power of 2 since $\mathrm{Stick}^S_{\mathcal{E}/F}(-1) = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{R} \supset 2^m\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$.

IV. Biquadratic Extensions

Now we assume that $\mathcal{E}/F$ is biquadratic, that is, $m = 2$. Denote the non-principal characters of G by χ_1, χ_2, χ_3, and the corresponding fields by $E_i = E_{\chi_i}$. Also let $e_i = e_{\chi_i}$ and $\tau_i = \tau_{\chi_i}$, restricting to the non-trivial automorphism of E_i/F. More specifically, we will take $\tau_1 = \tau_2$ fixing E_3 and τ_3 fixing E_1.

Proposition 4.1 ($\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ for a biquadratic extension not containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $F^{(1)} \not\subset \mathcal{E}$. Then*

$$\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E})) = \mathbb{Z}w_2(F)e_0 \oplus \bigoplus_{1\le i<j\le 3}\mathbb{Z}\big(w_2(E_i)^- e_i + w_2(E_j)^- e_j\big) = (\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}) \cap \mathcal{R},$$

of index 2 in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$

Proof. By Proposition 3.5,

$$\big(\mathrm{Ann}_{\mathcal{R}}W_2(\mathcal{E})\big)\mathcal{S} = \mathbb{Z}w_2(F)e_0 \oplus \bigoplus_i \mathbb{Z}w_2(E_i)^- e_i.$$

This contains $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, but cannot equal it, as it is not integral by Lemma 3.3. We will soon see that $2w_2(E_i)^- e_i \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$.

First let us show that $w_2(F)e_0 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. Here $e_0 = \frac{1}{4}(1+\tau_3)(1+\tau_1)$, as we have specified that τ_3 fix E_1. We have seen that $\sqrt{-1} \in W_2(F)$, and thus $4 \mid w_2(F)$. Hence we may integrally express $w_2(F)e_0 = (1+\tau_3)(1+\tau_1)(w_2(F)/4)$. Two applications of Proposition 3.6 then yield

$$\begin{aligned} W_2(\mathcal{E})^{e_0 w_2(F)} &= W_2(\mathcal{E})^{(1+\tau_3)(1+\tau_1)(w_2(F)/4)} \\ &= (W_2(E_1)^2)^{(1+\tau_1)(w_2(F)/4)} = W_2(E_1)^{(1+\tau_1)(w_2(F)/2)} \\ &= (W_2(F)^2)^{w_2(F)/2} = W_2(F)^{w_2(F)} = 1 \end{aligned}$$

This shows that $w_2(F)e_0 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. It should be clear that a similar computation using $4e_1 = (1+\tau_3)(1-\tau_1)$ will show that $2w_2(E_1)^- e_1 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, and by the same token, $2w_2(E_i)^- e_i \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$.

Finally we check that $w_2(E_1)^- e_1 + w_2(E_2)^- e_2 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. For this, note that we already know that twice this element lies in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. Hence it suffices to show that this element is integral and annihilates the 2-part of $W_2(\mathcal{E})$. We have taken $\tau_1 = \tau_2$ to be the unique non-trivial element fixing E_3, that is, lying in the kernel of χ_3. Since $w_2(E_1)^- \equiv w_2(E_1)^- \equiv 2 \pmod 4$ by Lemma 3.3, it follows easily that the element in question is a $\mathbb{Z}[G]$-multiple of $1 - \tau_1$. Then by Proposition 3.6 applied to $\mathcal{E}/E_3$, we have $W_2(\mathcal{E})^{1-\tau_1} = (W_2(\mathcal{E})_{1+\tau_1})^2$. Lemma 3.3 also implies that $4 \nmid |(W_2(\mathcal{E})_{1+\tau_1})|$. Thus the 2-part of $(W_2(\mathcal{E})_{1+\tau_1})^2$ is trivial, and this completes the check. Of course, by symmetry we find that the other elements obtained simply by permuting the subscripts of e_1, e_2 and e_3 lie in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ as well. Thus the direct sum in the statement of the proposition lies in $\mathrm{Ann}_R(W_2(\mathcal{E})) \subset (\mathrm{Ann}_R(W_2(\mathcal{E}))\mathcal{S}) \cap \mathcal{R} \neq \mathrm{Ann}_R(W_2(\mathcal{E}))\mathcal{S}$, and one can easily check that it has index 2 in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$. At the same time, $\big(\mathrm{Ann}_R(W_2(\mathcal{E}))\mathcal{S}\big) \cap \mathcal{R}$ has index at least 2 in $\mathrm{Ann}_R(W_2(\mathcal{E}))\mathcal{S}$, and the conclusion follows.

Corollary 4.2 ($\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ for a biquadratic extension not containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $F^{(1)} \not\subset \mathcal{E}$. Assume that the Birch-Tate conjecture holds for F and each relative quadratic extension of F in $\mathcal{E}$. Then*

$$\begin{aligned}\mathrm{Stick}^S_{\mathcal{E}/F}(-1) &\\ &= \mathbb{Z}k_2^S(F)e_0 \oplus \bigoplus_{1\le i<j\le 3} \mathbb{Z}\big(k_2^S(E_i)^- e_i + k_2^S(E_j)^- e_j\big)\\ &= (\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}) \cap (\theta^S_{\mathcal{E}/F}(-1)\mathcal{R}),\end{aligned}$$

of index 2 in $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$.

Proof. We begin with the equality in Proposition 4.2 and multiply by $\theta^S_{\mathcal{E}/F}(-1)$ to obtain a formula for $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$. Note that $\theta^S_{\mathcal{E}/F}(-1)$ is a non-zero divisor in $\mathbb{Q}[G]$ by Proposition 3.1. The result follows from Lemma 3.7. and the observation that the ambiguity in sign there affects the generators, but not the $\mathbb{Z}$-module they generate.

Remark 4.3. The idempotents in Corollary 4.2 have denominators equal to 4. However, known results such as [16, Cor. 6.3, Prop. 6.6] and Proposition 3.6 show that $k_2^S(F)$ and the $k_2^S(E_i)^-$ are multiples of 4.

For any prime number p, let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ denote the localization of $\mathbb{Z}$ at the prime ideal (p). Similarly, if M is a $\mathbb{Z}$-module, let $M_{(p)} \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ be the localization of M at (p). Note that $\mathcal{R}_{(p)} = \mathbb{Z}_{(p)}[G] \subset \mathbb{Q}[G]$ and the intersection of these over all primes p is $\mathcal{R}$. Also if I is an ideal of $\mathcal{R}$, then $I_{(p)} = I\mathcal{R}_{(p)}$ is an ideal of $\mathcal{R}_{(p)}$ and the intersection of these over all primes p is I.

Lemma 4.4. *If*

$$K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \neq K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_3)}$$

then $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *contains an element which is congruent to*

$$k_2^S(E_3)^- e_3 \equiv k_2^S(E_3)^- \frac{1+\tau_1}{2} \pmod{1+\tau_3}.$$

Proof. (Compare the proof of [15, Proposition 9.3]) Since $4\mathcal{S} \subset \mathcal{R}$, Proposition 3.9 implies that $4k_2^S(E_3)^- e_3 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$. Thus $k_2^S(E_3)^- e_3 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)_{(p)}$ for each odd prime p. It suffices to show that $\mathrm{Fit}^S_{\mathcal{E}/F}(1)_{(2)}$

contains an element congruent to $k_2^S(E_3)^- e_3 \equiv k_2^S(E_3)^- \frac{1+\tau_1}{2}$ modulo $(1+\tau_3)$. By the properties of Fitting ideals, this reduces to considering $M = K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_3}$ and showing that $k_2^S(E_3)^- \frac{1+\tau_1}{2}$ is in the Fitting ideal of $M_{(2)}$ over $\overline{\mathcal{R}}_{(2)} = \mathcal{R}_{(2)}/(1+\tau_3) \cong \mathbb{Z}_{(2)}[\langle\tau_1\rangle]$.

We claim that the cohomology group $\big(M_{(2)}\big)_{1+\tau_1}/M_{(2)}^{1-\tau_1}$ contains an element of order 2 under our hypothesis. We will establish this claim at the end of the proof. For now, we choose a minimal set of $\overline{\mathcal{R}}_{(2)}$-generators $\overline{\gamma}_i$ for $M_{(2)}/M_{(2)}^{1-\tau_1}$. Equivalently, these are a minimal set of generators over $\overline{\mathcal{R}}_{(2)}/(1-\tau_1) \cong \mathbb{Z}_{(2)}$, that is, generators for this finite abelian 2-group. We may assume (see [15, Lemma 9.2)]) that this group is the direct product of the subgroups of orders $d_i = 2^{c_i} > 1$ generated by the $\overline{\gamma}_i$, and that $\overline{\gamma}_1^{d_1/2} \in \big(M_{(2)}\big)_{1+\tau_1}/M_{(2)}^{1-\tau_1}$. Then $\prod_i d_i = |M_{(2)}/M_{(2)}^{1-\tau_1}|$, which is the 2-part of $|K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{(1-\tau_1,1+\tau_3)}| = |K_2(\mathcal{O}_{E_3}^S)/K_2(\mathcal{O}_{E_3}^S)^{1+\tau_3}| = |K_2(\mathcal{O}_{E_3}^S)_{1+\tau_3}| = k_2^S(E_3)^-$. Hence if we let D be the diagonal matrix of the d_i, then $\det(D)$ is associate to $k_2^S(E_3)^-$ in $\mathbb{Z}_{(2)}$. We can therefore complete the proof by showing that $\det(D)\frac{1+\tau_1}{2} \in \mathrm{Fit}_{\overline{\mathcal{R}}_{(2)}}(M_{(2)})$.

Now $\overline{\mathcal{R}}_{(2)}$ is a local ring and $1-\tau_1$ lies in the maximal ideal, so by Nakayama's lemma, the (arbitrarily chosen) inverse images $\gamma_i \in M_{(2)}$ of the $\overline{\gamma}_i$ generate $M_{(2)}$. We know that $\gamma_i^{d_i} \in M_{(2)}^{1-\tau_1}$; multiplying by $\gamma_i^{(-d_i/2)(1-\tau_1)}$ shows that $\gamma_i^{(d_i/2)(1+\tau_1)} \in M_{(2)}^{1-\tau_1}$, for each i. Hence there is a matrix B with entries in $\overline{\mathcal{R}}_2$ such that $D\big(\frac{1+\tau_1}{2}\big) - B\big(1-\tau_1\big)$ is a relations matrix for the generators γ_i of $M_{(2)}$. Furthermore, since $\gamma_1^{d_1/2} \in \big(M_{(2)}\big)_{1+\tau_1}$, we may choose the first row of B to be zero, so that $\det(B) =$

0. The very definition of the Fitting ideal then gives us that

$$\delta = \det\big(D(\frac{1+\tau_1}{2}) - B(1-\tau_1)\big) \in \mathrm{Fit}_{\overline{\mathcal{R}}_{(2)}}(M_{(2)}).$$

However,

$$\begin{aligned}\delta = \delta\big(\frac{1+\tau_1}{2} + \frac{1-\tau_1}{2}\big) &= \delta\frac{1+\tau_1}{2} + \delta\frac{1-\tau_1}{2}\\ &= \det\big(D(\frac{1+\tau_1}{2})\big) + \det\big(-2B\frac{1-\tau_1}{2}\big)\\ &= \det(D)\frac{1+\tau_1}{2} + \det(-2B)\frac{1-\tau_1}{2} = \det(D)\frac{1+\tau_1}{2}\end{aligned}$$

as desired.

Finally, we prove the claim. Indeed, we will see that $|\big(M_{(2)}\big)_{1+\tau_1}/M_{(2)}^{1-\tau_1}| = |K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1} \cap K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_3}/K_2(\mathcal{O}_{\mathcal{E}}^S)^{(1+\tau_1)(1+\tau_3)}|$. For this, note that the cohomology group $M_{1+\tau_1}/M^{1-\tau_1}$ over a group of order 2 must have exponent 2. Thus it is isomorphic to $\big(M_{(2)}\big)_{1+\tau_1}/M_{(2)}^{1-\tau_1}$. We now compute, using $K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{(1-\tau_1)} \cong K_2(\mathcal{O}_{E_3}^S)$ from Proposition 2.3.

$$\begin{aligned}|\big(M_{(2)}\big)_{1+\tau_1}/M_{(2)}^{1-\tau_1}| &= |M_{1+\tau_1}/M^{1-\tau_1}| = |M/M^{1-\tau_1}|/|M/M_{1+\tau_1}|\\ &= |K_2(\mathcal{O}_{\mathcal{E}}^S)/K_2(\mathcal{O}_{\mathcal{E}}^S)^{(1-\tau_1,1+\tau_3)}|/|M^{1+\tau_1}|\\ &= |K_2(\mathcal{O}_{E_3}^S)/K_2(\mathcal{O}_{E_3}^S)^{1+\tau_3}|/|K_2(\mathcal{O}_{\mathcal{E}}^S)^{(1+\tau_1,1+\tau_3)}/K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_3}|\\ &= |K_2(\mathcal{O}_{E_3}^S)_{1+\tau_3}|/|K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1}/(K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1} \cap K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_3})|\\ &= |K_2(\mathcal{O}_{E_3}^S)^-| \cdot |K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_3} \cap K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1}|/|K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1}|\end{aligned}$$

At this point, let $\epsilon(\mathcal{E}/E_3) = 1$ or 2, according to whether or not $\mathcal{E} = E_3^{(1)}$. Standard properties of the transfer $\mathrm{Tr}_{\mathcal{E}/E_3} : K_2(\mathcal{O}_{\mathcal{E}}^S) \to K_2(\mathcal{O}_{E_3}^S)$ and the natural map $\iota_{\mathcal{E}/E_3} : K_2(\mathcal{O}_{E_3}^S) \to K_2(\mathcal{O}_{\mathcal{E}}^S)$ induced by inclusion of rings imply that $K_2(\mathcal{O}_{\mathcal{E}}^S)^{1+\tau_1} = \iota_{\mathcal{E}/E_3}(\mathrm{Tr}_{\mathcal{E}/E_3}(K_2(\mathcal{O}_{\mathcal{E}}^S)))$. These maps are also

Galois-equivariant. In our case of totally real fields, $\mathrm{Tr}_{\mathcal{E}/E_3}(K_2(\mathcal{O}^S_{\mathcal{E}})) = K_2(\mathcal{O}^S_{E_3})$ by Proposition 2.3. We continue our computation with the use of these tools and Proposition 3.6.

$$\begin{aligned}
&|K_2(\mathcal{O}^S_{E_3})^-| \cdot |K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|/|K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}| \\
&= \epsilon(\mathcal{E}/E_3)|K_2(\mathcal{O}^S_{E_3})^-| \cdot |K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|/|K_2(\mathcal{O}_{E_3})| \\
&= \epsilon(\mathcal{E}/E_3)|K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|/|K_2(\mathcal{O}_{E_3})^{1+\tau_3}| \\
&= \frac{|K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|}{|K_2(\mathcal{O}^S_{E})^{(1+\tau_1)(1+\tau_3)}|} \cdot \frac{\epsilon(\mathcal{E}/E_3)|(\iota_{\mathcal{E}/E_3}(K_2(\mathcal{O}^S_{E_3})))^{1+\tau_3}|}{|K_2(\mathcal{O}^S_{E_3})^{1+\tau_3}|} \\
&= \frac{|K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|}{|K_2(\mathcal{O}^S_{E})^{(1+\tau_1)(1+\tau_3)}|} \cdot \frac{\epsilon(\mathcal{E}/E_3)|\iota_{\mathcal{E}/E_3}((K_2(\mathcal{O}^S_{E_3}))^{1+\tau_3})|}{|K_2(\mathcal{O}^S_{E_3})^{1+\tau_3}|} \\
&= \frac{|K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1}|}{|K_2(\mathcal{O}^S_{E})^{(1+\tau_1)(1+\tau_3)}|} \cdot \frac{\epsilon(\mathcal{E}/E_3)}{|\ker(\iota_{\mathcal{E}/E_3}|_{K_2(\mathcal{O}^S_{E_3})^{1+\tau_3}})|}
\end{aligned}$$

We now show that the second fraction here is in fact equal to 1. So consider $|\ker(\iota_{\mathcal{E}/E_3}|_{K_2(\mathcal{O}^S_{E_3})^{1+\tau_3}})| = |\ker(\iota_{\mathcal{E}/E_3}|_{\iota_{E_3/F}(K_2(\mathcal{O}^S_F))})|$. From Proposition 4.1, we know that $|\ker(\iota_{\mathcal{E}/E_3})| = \epsilon(\mathcal{E}/E_3) \leq 2$. When $\epsilon(\mathcal{E}/E_3) = 1$ it is clear that $|\ker(\iota_{\mathcal{E}/E_3}|_{\iota_{E_3/F}(K_2(\mathcal{O}^S_F))})| = 1 = \epsilon(\mathcal{E}/E_3)$. When $\epsilon(\mathcal{E}/E_3) = 2$, Proposition 7.1 of [15] shows that a non-trivial element of $\ker(\iota_{\mathcal{E}/E_3})$ is given by $\{-1, a\}_{E_3}$, for $a \in E_3$ such that $\mathcal{E} = E_3(\sqrt{a})$. Since $\mathcal{E}/F$ is biquadratic, we may choose $a \in F$. Then $\{-1, a\}_{E_3} = \iota_{E_3/F}(\{-1, a\}_F) \in \ker(\iota_{\mathcal{E}/E_3}|_{\iota_{E_3/F}(K_2(\mathcal{O}^S_F))})$, which must then be of order $2 = \epsilon(\mathcal{E}/E_3)$.

Theorem 4.5 (Comparison Theorem for a biquadratic extension not containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $\mathcal{E}$ does not contain $F^{(1)}$. Then* $\mathrm{Fit}^S_{\mathcal{E}/F}(1) \supset 2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = 2\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$. *If* $K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \neq K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_3)}$ *and* $K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1\tau_3} \neq K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_1\tau_3)}$, *then either* *a.* $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S} \supset \mathrm{Stick}^S_{\mathcal{E}/F}(-1)$, *or*

b. $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *has index 2 in* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$. *Under the assumption of the Birch-Tate conjecture for* F *and the* E_i, $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *and* $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ *then have the same index in* $\mathcal{R}$.

Proof. By Proposition 3.9, $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$ is generated by $|K_2(\mathcal{O}^S_F)|e_0$ and the $k^S_2(E_i)^- e_i$. We show that twice each of these elements lies in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$.

Put $\overline{G}_3 = \mathrm{Gal}(E_3/F)$ and consider the projection π_3 from $\mathcal{R} = \mathbb{Z}[G]$ to $\overline{\mathcal{R}} = \mathbb{Z}[\overline{G}_3]$, with kernel generated by $1-\tau_1$. By Proposition 2.4, $\pi_3(\mathrm{Fit}^S_{\mathcal{E}/F}(1)) = \mathrm{Fit}^S_{E_3/F}(1)$. By Proposition 1.3 with $E_3 \neq F^{(1)}$, $k^S_2(F)\frac{1+\tau_3}{2}$ and $k^S_2(E_3)^-\frac{1-\tau_3}{2}$ both lie in $\mathrm{Fit}^S_{E_3/F}(1)$. Thus $k^S_2(F)\frac{1+\tau_3}{2} + (1-\tau_1)\rho_1$ and $k_2(E_3)^-\frac{1-\tau_3}{2} + (1-\tau_1)\rho_2$ lie in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$, for some ρ_1 and ρ_2 in $\mathcal{R}$. Multiplying by $1+\tau_1$ in $\mathcal{R}$, we deduce that $2k^S_2(F)e_0$ and $2k_2(E_3)^- e_3$ lie in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$. A similar argument shows that $2k^S_2(E_2)^- e_2 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$ and $2k^S_2(E_1)^- e_1 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$. It follows that $2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} \subset \mathrm{Fit}^S_{\mathcal{E}/F}(1)$.

Thus we may consider $\mathrm{Fit}^S_{\mathcal{E}/F}(1)/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$ as an $\mathbb{F}_2$-subspace of the 4-dimensional space $(\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$. If it has dimension 4, then $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$, and this is case (a). If it has dimension 3, then $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ clearly has index 2 in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$. This is case (b). Under the assumption of the Birch-Tate conjecture, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ also has index 2 in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$, by Corollary 4.2 and Theorem 3.10. It follows that $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ have the same index in $\mathcal{R}$.

Under our additional assumptions, we now show by contradiction that $V = \mathrm{Fit}^S_{\mathcal{E}/F}(1)/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$ cannot have dimension less than 3. For $1 \leq i \leq 3$ let $\overline{G}_i = \mathrm{Gal}(E_i/F)$ and let $\mathcal{S}_i$ denote the maximal order in $\mathbb{Q}[\overline{G}_i]$. Propositions 2.4 and 1.3 imply that, for each i from 1 to 3 inclusive, V projects onto $\mathrm{Fit}^S_{E_i/F}(1)/\big(2\mathrm{Fit}^S_{E_i/F}(1)\mathcal{S}_i\big)$, of dimension 2, generated by the images of $k_2^S(F)e_0$ and $k_2^S(E_i)^- e_i$. So V has dimension at least 2, hence exactly 2, and the projection is an isomorphism. Then V must contain exactly one non-trivial element whose e_0-component is 0, and the three projections show that this element must be $v_1 = k_2^S(E_1)^- e_1 + k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3$. The subspace V also contains an element with a non-trivial e_0-component, and by adding v_1 if necessary, we see that V contains $v_2 = k_2^S(F)e_0 + k_2^S(E_i)^- e_i$ for some i. For σ_i generating $\ker(\chi_i)$ we now take images modulo $1+\sigma_i$, which amounts to projecting onto the two-dimensional space spanned by the e_j for $j \neq 0, i$. Thus the image of v_2 modulo $1+\sigma_i$ is 0. The image of v_1 modulo $1+\sigma_i$ has two non-zero components, and thus the image of V modulo $1+\sigma_i$ does not contain an element with exactly one non-zero component. This contradicts Lemma 4.4.

Proposition 4.6 **(**$\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ **for a biquadratic extension containing** $F^{(1)}$**).** *Suppose that* $\mathcal{E}/F$ *is biquadratic, and that* $E_1 = F^{(1)} \subset \mathcal{E}$*. Then*

$$\begin{aligned}\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E})) = \mathbb{Z}\big(w_2(F)e_0 + w_2(E_1)^- e_1 + w_2(E_2)^- e_2\big)\\ \oplus \mathbb{Z}\big(w_2(E_2)^- e_2 + w_2(E_3)^- e_3\big)\\ \oplus \mathbb{Z}2w_2(F)e_0 \oplus \mathbb{Z}2w_2(E_1)^- e_1\end{aligned}$$

of index 4 in $\big(\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\big)\mathcal{S}$

Proof. As above, we take τ_3 to fix E_1, and $\tau_1 = \tau_2$ to fix E_3. By Proposition 3.5,

$$\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S} = \mathbb{Z}w_2(F)e_0 \oplus \bigoplus_i \mathbb{Z}w_2(E_i)^- e_i.$$

As in the proof of Proposition 4.1, this contains, but does not equal $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, since $w_2(E_i)^- \equiv 2 \pmod 4$ for each i, by Lemma 3.3. This time, two applications of Proposition 3.6 yield

$$\begin{aligned} W_2(\mathcal{E})^{e_0 w_2(F)} &= W_2(\mathcal{E})^{(1+\tau_3)(1+\tau_1)(w_2(F)/4)} \\ &= (W_2(E_1)^2)^{(1+\tau_1)(w_2(F)/4)} = W_2(E_1)^{(1+\tau_1)(w_2(F)/2)} \\ &= W_2(F)^{w_2(F)/2} = \{\pm 1\}. \end{aligned}$$

Now we can see that $w_2(F)e_0 \notin \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, but $2w_2(F)e_0 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. Similarly, $2w_2(E_i)^- e_i \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, but $w_2(E_i)^- e_i$ is not integral, so does not lie in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. So far, we know that $2\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S} \subset \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E})) \subset \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$. The proof that $w_2(E_2)^- e_2 + w_2(E_3)^- e_3 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ goes just as in Proposition 4.2, since we have observed that $F^{(2)} \not\subset \mathcal{E}$ and thus $\mathcal{E}$ is not the first layer of the cyclotomic $\mathbb{Z}_2$-extension of $E_1 = F^{(1)}$. Finally, to see that $w_2(F)e_0 + w_2(E_1)^- e_1 + w_2(E_2)^- e_2 \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, note that again we know that twice this element lies in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$, and thus is suffices to see that this element annihilates the 2-part of $W_2(\mathcal{E})$. Indeed, as $2w_2(E_i)^- e_i \in \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ and $2w_2(E_i)^-$ is 4 times an odd number for $i \geq 1$, we conclude that $4e_i$ annihilates the 2-part of $W_2(\mathcal{E})$ for $i \geq 1$. After reducing modulo these known annihilators of the 2-part of $W_2(\mathcal{E})$, it suffices to show that $w_2(F)e_0 + 2e_1 + 2e_2 = w_2(F)e_0 + (1 - \tau_1)$ annihilates the 2-part of $W_2(\mathcal{E})$. By Proposition 3.6 and Lemma 3.3 again, $W_2(\mathcal{E})^{1-\tau_1} = W_2(\mathcal{E})_{1+\tau_1}$, whose 2-part is $\{\pm 1\}$.

Thus if ω is a generator for the 2-part of $W_2(\mathcal{E})$, then $\omega^{1-\tau_1} = -1$. At the same time, we know that $\omega^{e_0 w_2(F)} = -1$ since $W_2(\mathcal{E})^{e_0 w_2(F)} = \{\pm 1\}$. Combining these shows that $\omega^{w_2(F)e_0+(1-\tau_1)} = 1$, as desired.

It is easy to check that the ideal in the statement of the Proposition is contained in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ and has index 4 in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$. If $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$ strictly contains this ideal, it must be of index 1 or 2 in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$ and lie in $\mathcal{R}$. So indeed it must equal $\big(\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}\big) \cap \mathcal{R}$, which is of index 2 in $\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}$ as in Proposition 4.1. However $w_2(F)e_0 \in \big(\mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))\mathcal{S}\big) \cap \mathcal{R}$, while we have seen that $w_2(F)e_0 \notin \mathrm{Ann}_{\mathcal{R}}(W_2(\mathcal{E}))$. The conclusion follows.

Corollary 4.7 ($\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ **for a biquadratic extension containing** $F^{(1)}$)**.** *Suppose that $\mathcal{E}/F$ is biquadratic, and that $E_1 = F^{(1)}$. Assume that the Birch-Tate conjecture holds for F and each relative quadratic extension of F in $\mathcal{E}$. Then*

$$\mathrm{Stick}^S_{\mathcal{E}/F}(-1) = \mathbb{Z}\big(k_2^S(F)e_0 + k_2^S(E_1)^- e_1 + k_2^S(E_2)^- e_2\big) \\ \oplus \mathbb{Z}\big(k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3\big) \oplus \mathbb{Z}2k_2(F)e_0 \oplus \mathbb{Z}2k_2^S(E_1)^- e_1$$

of index 4 in $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$

Proof. We begin with the equality in Proposition 4.6 and multiply by $\theta^S_{\mathcal{E}/F}(-1)$ to obtain a formula for $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$. Again, $\theta^S_{\mathcal{E}/F}(-1)$ is a non-zero divisor in $\mathbb{Q}[G]$ by Proposition 3.1. The result follows from Lemma 3.7 and the observation that the choice of signs there does not affect the $\mathbb{Z}$-module generated.

Theorem 4.8 (Comparison Theorem for a biquadratic extension containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $E_1 = F^{(1)}$. Then $\mathrm{Fit}^S_{\mathcal{E}/F}(1) \supset 2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = 2\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$ must be one of three $\mathbb{F}_2$-subspaces of $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$ (We cannot say that all three occur). The bases for these subspaces are:*

a. $\{k_2^S(F)e_0 + k_2^S(E_1)^- e_1,\ k_2^S(E_2)^- e_2,\ k_2^S(E_3)^- e_3\}$

b. $\{k_2^S(F)e_0 + k_2^S(E_1)^- e_1 + k_2^S(E_2)^- e_2,\ k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3\}$

c. $\{k_2^S(F)e_0 + k_2^S(E_1)^- e_1,\ k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3\}$

Now assume that the Birch-Tate conjecture holds for F and each E_i. If case (a) occurs, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ lies in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ with index 2. If case (b) occurs, $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)$. If case (c) occurs, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ have the same index in $\mathcal{R}$. If $K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \neq K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_3)}$, then case (c) does not occur.

Proof. Again by Proposition 3.9, $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$ is generated by $|K_2(\mathcal{O}^S_F)|e_0$ and the $k_2^S(E_i)^- e_i$, and we show that twice each of these elements lies in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$. The proof that $2k_2^S(F)e_0$, $2k_2(E_3)^- e_3$ and $2k_2^S(E_2)^- e_2$ lie in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ goes just as in Theorem 4.5.

For $E_1 = F^{(1)}$, Proposition 1.3 gives $k_2^S(F)\frac{1+\tau_1}{2} + k_2^S(E_1)^- \frac{1-\tau_1}{2} \in \mathrm{Stick}^S_{E_1/F}(-1)$. We obtain $k_2^S(F)\frac{1+\tau_1}{2} + k_2^S(E_1)^- \frac{1-\tau_1}{2} + (1-\tau_3)\rho_3 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$, and multiplication by $1+\tau_3$ gives $2k_2^S(F)e_0 + 2k_2^S(E_1)^- e_1 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$. As we already know that $2k_2^S(F)e_0 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$, we conclude that $2k_2^S(E_1)^- e_1 \in \mathrm{Fit}^S_{\mathcal{E}/F}(1)$

We now consider the images of the $\mathbb{F}_2$-subspace $\mathrm{Fit}^S_{\mathcal{E}/F}(1)/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$

of the 4-dimensional space $(\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})/(2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S})$ under projection onto certain 2-dimensional subspaces. So the kernels of these projections are 2-dimensional. Let $\mathcal{S}_3$ denote the maximal order in $\mathbb{Q}[\overline{G}_3]$. According to Proposition 2.4 and Proposition 1.3, projecting via π_3 maps the subspace onto $\mathrm{Fit}^S_{E_3/F}(1)/\big(2\mathrm{Fit}^S_{E_3/F}(1)\mathcal{S}_3\big)$, of dimension 2, generated by the images of $|K_2(\mathcal{O}_F^S)|e_0$ and $|K_2(\mathcal{O}^S_{E_3})_{1+\tau_3}|e_3$. So our subspace in question has dimension at least 2. On the other hand, projecting via π_1 yields a one-dimensional image spanned by the image of $k_2^S(F)e_0 + k_2^S(E_1)^- e_1$, according to Proposition 2.4 and Proposition 1.3, since $E_1 = F^{(1)}$. The kernel has dimension at most 2. So if our subspace has dimension 3, it must be that it contains the kernel of the projection induced by π_1, as well as $k_2^S(F)e_0 + k_2^S(E_1)^- e_1$. This results in case (a).

If our subspace is 2-dimensional, it contains just one nontrivial element of the kernel of the projection induced by π_1. If this element is not $k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3$, then the projection of this element via π_2 or π_3 will be 0 and the image of our subspace will be 1-dimensional. We have already seen that this is not the case for π_3, and hence likewise for π_2. So our subspace must contain $k_2^S(E_2)^- e_2 + k_2^S(E_3)^- e_3$, which is congruent to 0 modulo $1-\tau_3$, and it must also contain an element congruent to $k_2^S(F)e_0 + k_2^S(E_1)^- e_1$ modulo $1-\tau_3$. This leaves only cases (b) and (c).

The statements concerning $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ are now clear from Corollary 4.7. Proof of the final claim follows from Lemma 4.4 (and Remark 4.3), since $k_2^S(F)e_0 \equiv k_2({}^S(E_1)^- e_1 \equiv 0$ modulo $1+\tau_3$, while $k_2^S(E_2)^- e_2 \equiv k_2^S(E_2)^- \frac{1-\tau_1}{2}$.

Proposition 4.9 (The index of the higher Stickelberger ideal for a biquadratic extension). *Assume that $\mathcal{E}/F$ is biquadratic, and that the Birch-Tate conjecture holds for $\mathcal{E}$, F, and the intermediate fields. Then*

$$(\mathcal{R} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1)) = |K_2(\mathcal{O}^S_{\mathcal{E}})|.$$

Proof. By Proposition 3.14

$$(\mathcal{R} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1)) = |K_2(\mathcal{O}^S_{\mathcal{E}})| \frac{(\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1))}{2\delta}.$$

At the same time, Corollaries 4.2 and 4.7 give $(\mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S} : \mathrm{Stick}^S_{\mathcal{E}/F}(-1)) = 2\delta$.

Remark 4.10 (The index of the higher Stickelberger ideal for a quadratic extension). The result of Proposition 4.9 holds for a relative quadratic extension $\mathcal{E}/F$ as well. This follows from Proposition 1.3. Still it seems that factors of 2 may intervene in larger multi-quadratic extensions.

V. Applications

For easy reference, we first record some standard facts in a Lemma.

Lemma 5.1. *Suppose that E/F is a relative quadratic extension and that α and β lie in $E^\times$. Then*

1. *$E(\sqrt{\alpha}) = E(\sqrt{\beta})$ if and only if $\alpha\beta$ is a square in E.*
2. *$E(\sqrt{\alpha})/F$ is a Galois extension if and only if the relative norm of α is a square in E.*
3. *$E(\sqrt{\alpha})/F$ is a biquadratic extension if and only if α is not a square in E and the relative norm of α is a square in F.*

Proof.

1. This follows from Kummer theory or an easy exercise.
2. This follows from (1) upon taking β to be the conjugate of α over F.
3. Suppose that the extension is biquadratic. Then $E(\sqrt{\alpha}) = E(\sqrt{a})$ for some $a \in F$. Apply (1) and take the norm. For the converse, let c^2 be the norm of α. The automorphisms sending $\sqrt{\alpha}$ to its conjugates $\pm c/\sqrt{\alpha}$ both have order two, so cannot lie in a cyclic group.

Proposition 5.2. *Let $F = \mathbb{Q}$ and let E_1 be a real quadratic field of discriminant d for which the prime divisors q_j of d are not congruent to 1 modulo 4. Let r be a positive, non-square integer which is a norm from $\mathcal{O}_{E_1}$, and whose prime divisors p_i are also not congruent to 1 modulo 4. Assume further that rd is not a square. Let $E_3 = \mathbb{Q}(\sqrt{r})$, and let S contain $\{\infty\} \cup \{q_1,\ q_2, \dots\} \cup \{p_1,\ p_2, \dots\}$, but no finite primes congruent to 1 modulo 4. Then for $\mathcal{E} = \mathbb{Q}(\sqrt{d}, \sqrt{r})$, we have $K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} \neq K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_3)}$.*

Proof. Let $\alpha \in \mathcal{O}_{E_1}$ have norm r. We claim that the element $\{-1, \alpha\}_{\mathcal{E}} \in K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_1} \cap K_2(\mathcal{O}^S_{\mathcal{E}})^{1+\tau_3} = \iota_{\mathcal{E}/E_3}(K_2(\mathcal{O}^S_{E_3})) \cap \iota_{\mathcal{E}/E_1}(K_2(\mathcal{O}^S_{E_1}))$, and $\{-1, \alpha\}_{\mathcal{E}} \notin K_2(\mathcal{O}^S_{\mathcal{E}})^{(1+\tau_1)(1+\tau_3)} = \iota_{\mathcal{E}/F}(K_2(\mathcal{O}^S_F))$. (These equalities are seen in the proof of Lemma 4.4.) First, since $\alpha \in \mathcal{O}_{E_1}$ and $N(\alpha) = r$ is an S-unit, α is also an S-unit. Therefore $\{-1, \alpha\}_{E_1} \in K_2(\mathcal{O}^S_{E_1})$ and clearly $\{-1, \alpha\}_{\mathcal{E}} = \iota_{\mathcal{E}/E_1}\{-1, \alpha\}_{E_1} \in \iota_{\mathcal{E}/E_1}(K_2(\mathcal{O}^S_{E_1}))$.

To see that $\{-1, \alpha\}_{\mathcal{E}} \in \iota_{\mathcal{E}/E_3}(K_2(\mathcal{O}^S_{E_3}))$, consider the extension $\mathcal{E}(\sqrt{\alpha})/E_3$. The relative norm of α in $\mathcal{E}/E_3$ is r, which is a square in E_3. So this extension is biquadratic by Lemma 5.1, and we have $\mathcal{E}(\sqrt{\alpha}) = \mathcal{E}(\sqrt{\beta})$ for some $\beta \in E_3$. Also by Lemma 5.1, $\alpha\beta$ is a square in $\mathcal{E}$. It follows that $\{-1, \alpha\}_{\mathcal{E}} = \{-1, \beta\}_{\mathcal{E}} = \iota_{\mathcal{E}/E_3}(\{-1, \beta\}_{E_3}) \in \iota_{\mathcal{E}/E_3}(K_2(\mathcal{O}^S_{E_3}))$. Note that $\mathcal{E}(\sqrt{\beta})/\mathbb{Q}$ is unramified outside $S \cup \{2\}$, and this ensures that $\{-1, \beta\}_{E_3} \in K_2(E_3)$ actually lies in the S-tame kernel $K_2(\mathcal{O}^S_{E_3})$ (see [16, Proposition 6.1]).

Now we suppose that $\{-1, \alpha\}_{\mathcal{E}} \in \iota_{\mathcal{E}/\mathbb{Q}}(K_2(\mathbb{Z}^S))$ and derive a contradiction. Being of order 2, this element must be the image of an element of 2-power order in $K_2(\mathbb{Z}^S)$. Our choice of S ensures that there are no elements of order 4 in $K_2(\mathbb{Z}^S)$. For it follows from Tate's computation of $K_2(\mathbb{Q})$ (see [12, Section 11]) that $K_2(\mathbb{Z}^S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\infty \neq p \in S} (\mathbb{Z}/pZ)^\times$. Thus $\{-1, \alpha\}_{\mathcal{E}}$ must be the image of an element of order 2 in $K_2(\mathbb{Z}^S)$, and such elements are of the form $\{-1, a\}_{\mathbb{Q}}$ for some $a \in \mathbb{Q}$, by [21, Theorem 6.1]. We must have $\{-1, \alpha\}_{\mathcal{E}} = \{-1, a\}_{\mathcal{E}}$, or $\{-1, \alpha/a\}_{\mathcal{E}} = 1$. In other words, α/a is in the Tate kernel. For the totally real field $\mathcal{E}$, the Tate kernel is generated by $(\mathcal{E}^\times)^2$ and an element $\pi \in \mathcal{E}$ for which $\sqrt{\pi}$ lies in the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$ (see [10,Proposition 2.4]). Hence $\alpha = a\pi^m\gamma^2$ for some integer m and element $\gamma \in \mathcal{E}$. So $\mathcal{E}(\sqrt{\alpha}) \subset \mathcal{E}(\sqrt{a}, \sqrt{\pi})$, which is an abelian Galois extension of $\mathbb{Q}$. Consequently, the subfield $E_1(\sqrt{\alpha})$ is also an abelian Galois extension of $\mathbb{Q}$. On the other hand, the norm of α is r, which is not a square in $\mathbb{Q}$. Furthermore, r is not a square in E_1 for otherwise $E_1 = \mathbb{Q}(\sqrt{r})$ and rd would be a square in $\mathbb{Q}$ by Lemma 5.1 (1) again. Then by Lemma 5.1 (2), $E_1(\sqrt{\alpha})$ is not a Galois extension of $\mathbb{Q}$, and this is a contradiction.

Corollary 5.3. *Let r be a product of one or more distinct primes which are congruent to -1 modulo 8, or twice such a product. Let S contain ∞, 2, and the prime divisors of r, but no finite prime congruent to 1 modulo 4. Then for $F = \mathbb{Q}$ and $\mathcal{E} = \mathbb{Q}(\sqrt{2}, \sqrt{r})$, we have*

$$\mathrm{Stick}^S_{\mathcal{E}/F}(-1) \subset \mathrm{Fit}^S_{\mathcal{E}/F}(1),$$

and the index is 1 or 2.

Proof. This follows from Proposition 1.2, Proposition 5.2 and Proposition 4.8.

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