# VALUES AT $s = -1$ OF $L$-FUNCTIONS FOR RELATIVE QUADRATIC EXTENSIONS OF NUMBER FIELDS, AND THE FITTING IDEAL OF THE TAME KERNEL

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Abstract. Fix a relative quadratic extension $E/F$ of totally real number fields and let $G$ denote the Galois group of order 2. Let $S$ be a finite set of primes of $F$ containing the infinite primes and all those which ramify in $E$, let $S_E$ denote the primes of $E$ lying above those in $S$, and let $\mathcal{O}_E^S$ denote the ring of $S_E$-integers of $E$. Assume the truth of the 2-part of the Birch-Tate conjecture relating the order of the tame kernel $K_2(\mathcal{O}_E)$ to the value of the Dedekind zeta function of $E$ at $s = -1$, and assume the same for $F$ as well. We then prove that the Fitting ideal of $K_2(\mathcal{O}_E^S)$ as a $\mathbb{Z}[G]$-module is equal to a generalized Stickelberger ideal. Equality after tensoring with $\mathbb{Z}[1/2][G]$ holds unconditionally.

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## I. Introduction

Let $E/F$ be a fixed relative quadratic extension of algebraic number fields with Galois group $G = \langle \tau \rangle$. We also fix a finite set $S$ of primes of $F$ which contains all of the infinite primes of $F$ and all of the primes which ramify in $E$. Associated with this data is a Stickelberger function, or equivariant $L$-function, $\theta^S_{E/F}(s)$. It is a meromorphic function of $s$ with values in the group ring $\mathbb{C}[G]$. To define it, let $\mathfrak{p}$ run through the (finite) primes of $F$ not in $S$, and $\mathfrak{a}$ run through integral ideals of $F$ which are relatively prime to each of the elements of $S$. Then $\mathrm{N}\mathfrak{a}$ denotes the absolute norm of the ideal $\mathfrak{a}$, $\sigma_{\mathfrak{a}} \in G$ is the well-defined automorphism attached to $\mathfrak{a}$ via the Artin map, and

$$\theta^S_{E/F}(s) = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a},S)=1}} \frac{1}{\mathrm{N}\mathfrak{a}^s} \sigma_{\mathfrak{a}}^{-1} = \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s} \sigma_{\mathfrak{p}}^{-1}\right)^{-1}.$$

These expressions converge for the real part of $s$ greater than 1 and the function they define extends meromorphically to all of $\mathbb{C}$.

The function $\theta^S_{E/F}(s)$ is connected with the arithmetic of the number fields $E$ and $F$ in ways one would like to make as precise as possible. Define the ring of $S$-integers $\mathcal{O}^S_F$ of $F$ to be the set of elements of $F$ whose valuation is non-negative at every prime not in $S$. Similarly, define the ring $\mathcal{O}^S_E$ of $S$-integers of $E$ to be the set of elements of $E$ whose valuation is non-negative at every prime not in $S_E$, the set of all primes of $E$ which lie above some prime in $S$. The zeta-functions defined by $\zeta^S_F(s)\sigma_{\mathcal{O}_F} = \theta^S_{F/F}(s)$ and $\zeta^S_E(s)\sigma_{\mathcal{O}_E} = \theta^{S_E}_{E/E}(s)$ may be viewed as the zeta-functions of the Dedekind domains $\mathcal{O}_F^S$ and $\mathcal{O}_E^S$.

Our focus will be on the "higher Stickelberger element" $\theta_{E/F}^S(-1)$. It is conjecturally related to the algebraic $K$-group $K_2(\mathcal{O}_E^S)$. This group is known to be finite by [3] and [10], and called the tame kernel of $E$.

The precise statement of the conjectured arithmetic interpretation of $\theta_{E/F}^S(-1)$ will also involve another finite group. Let $\mu_\infty$ denote the group of all roots of unity in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing $E$, and let $\mathcal{G}$ denote the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. Define $W_2 = W_2(\overline{\mathbb{Q}})$ to be the $\mathbb{Z}[\mathcal{G}]$-module whose underlying group is $\mu_\infty$, with the action of $\gamma \in \mathcal{G}$ on $\omega \in W_2$ given by $\omega^\gamma = \gamma^2(\omega)$. Then for any subfield $L$ of $\overline{\mathbb{Q}}$, let $W_2(L)$ be the submodule fixed under this action by the Galois group of $\overline{\mathbb{Q}}$ over $L$. Then $W_2(E)$ naturally becomes a $\mathbb{Z}[G]$-module, where the action of $G$ arises by lifting elements of $G$ to the Galois group of $\overline{\mathbb{Q}}$ over $F$. One easily sees that the $G$-fixed submodule $W_2(E)^G$ equals $W_2(F)$. We use the notation $w_2(L) = |W_2(L)|$, which we note is finite for any algebraic number field $L$.

We are now ready to state the conjecture of Birch and Tate (see section 4 of [13]), which gives a precise arithmetic interpretation of $\zeta_F^S(-1)$. We state an extended form of it for arbitary finite $S$ which is an easy consequence of the original conjecture for minimal $S$ (see Corollary 3.3 of [11]).

**Conjecture 1.1 (Extended Birch-Tate).** *Suppose that $F$ is totally real. Then*

$$\zeta_F^S(-1) = (-1)^{|S|} \frac{|K_2(\mathcal{O}_E^S)|}{w_2(F)}$$

Deep results on Iwasawa's Main conjecture in [9] and [14] lead to the following (see [7]).

**Theorem 1.2.** *The Birch-Tate Conjecture holds if $F$ is abelian over $\mathbb{Q}$, and the odd part holds for all totally real $F$.*

We note that the 2-part of the Birch-Tate conjecture for $F$ would follow from the 2-part of Iwasawa's Main conjecture for $F$. We can now state our main result, using the annihilator $\mathrm{Ann}_{\mathbb{Z}[G]}(W_2(E))$ of $W_2(E)$ in $\mathbb{Z}[G]$.

**Theorem 1.3.** *Let $E/F$ be a relative quadratic extension of totally real number fields, with Galois group $G$. Assume that the 2-part of the Birch-Tate conjecture holds for $E$ and for $F$. Then the (first) Fitting ideal of $K_2(\mathcal{O}_E^S)$ as a $\mathbb{Z}[G]$-module is*

$$\mathrm{Fit}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_E^S)) = \mathrm{Ann}_{\mathbb{Z}[G]}(W_2(E))\theta_{E/F}^S(-1).$$

*Equality of the extension of these ideals to $\mathbb{Z}[1/2][G]$ holds unconditionally.*

This result is to be compared with results computing the Fitting ideals of certain modified ideal class groups in terms of more classical Stickelberger ideals. Notable among these is the recent result of Greither [5] identifying the Fitting ideal of the dual of the odd part of the minus part of the ideal class group of a CM field which is abelian over a totally real field. Assuming the Equivariant Tamagawa Number Conjecture, this ideal is shown to equal a Stickelberger ideal obtained from the values at $s = 0$ of Stickelberger functions for subextensions. Our Theorem 1.3 begins to provide an analog at $s=-1$. The comparison can provide some insight since the odd part in Theorem 1.3 is unconditional, and the result on the 2-part, though conditional, has no counterpart as yet at $s=0$.

For ease of notation from now on, let $\mathcal{R}=\mathbb{Z}[G]$. Denote the Fitting ideal we are interested in by $\mathcal{I}^{\mathrm{Fit}}=\mathrm{Fit}_{\mathcal{R}}(K_2(\mathcal{O}_E^S))$, and the generalized Stickelberger ideal by $\mathcal{I}^{\mathrm{Sti}}=\mathrm{Ann}_{\mathcal{R}}(W_2(E))\theta_{E/F}^S(-1)$. The ingredients we will need to prove their equality are cohomology computations, properties of Fitting ideals, and some information on the ideal structure of $\mathcal{R}$ and related rings.

## II. Cohomology of $K_2(\mathcal{O}_E^S)$

We continue to assume throughout that $E/F$ is a relative quadratic extension of totally real fields with Galois group $G=\langle\tau\rangle$. For a $\mathbb{Z}[G]$-module $M$, we define $M^G$ as usual to be the submodule of $M$ which is annihilated by $1-\tau$, and $M^-$ to be the submodule annihilated by $1+\tau$. Also let $M_G$ denote the module of co-invariants of $M$ under the action of $G$. The following result provides the key to computing the orders we will need.

**Theorem 2.1.** *Under our assumptions, the transfer map from $K_2(\mathcal{O}_E^S)$ to $K_2(\mathcal{O}_F^S)$ is surjective with kernel $K_2(\mathcal{O}_E^S)^{1-\tau}$. So we have a short exact sequence*

$$0\to K_2(\mathcal{O}_E^S)^{1-\tau}\to K_2(\mathcal{O}_E^S)\overset{\mathrm{Trans}}{\to} K_2(\mathcal{O}_F^S)\to 0$$

*Proof.* In the more general setting of a Galois extension of number fields $E/F$, with Galois group $G$, Kahn's theorem 5.1 of [6] leads to an exact sequence induced by the transfer map and involving the number $r_\infty(E/F)$ of infinite primes of $F$ which ramify in $E$ (see [8, Prop. 1.6] or [11, Thm. 3.4]):

$$0 \to K_2(\mathcal{O}_E^S)_G \to K_2(\mathcal{O}_F^S) \to \{\pm 1\}^{r_\infty(E/F)} \to 0$$

In our situation with $G = \langle \tau \rangle$ cyclic and $E$ totally real, $K_2(\mathcal{O}_E^S)_G = K_2(\mathcal{O}_E^S)/K_2(\mathcal{O}_E^S)^{1-\tau}$ and $r_\infty(E/F) = 0$. The result follows. □

**Proposition 2.2.** $|K_2(\mathcal{O}_E^S)^G| = |K_2(\mathcal{O}_F^S)|$

*Proof.* Theorem 2.1 gives $|K_2(\mathcal{O}_F^S)| = |K_2(\mathcal{O}_E^S)|/|K_2(\mathcal{O}_E^S)^{1-\tau}|$, while the exact sequence

$$0 \to K_2(\mathcal{O}_E^S)^G \to K_2(\mathcal{O}_E^S) \stackrel{1-\tau}{\to} K_2(\mathcal{O}_E^S)^{1-\tau} \to 0$$

gives the same value for $|K_2(\mathcal{O}_E^S)^G|$. □

Let $\mathbb{Q}_r$ denote the $r$th layer of the cyclotomic $\mathbb{Z}_2$ extension of $\mathbb{Q}$; it may be defined as $\mathbb{Q}_r = \mathbb{Q}(\mu_{2^{r+2}})^+$, the maximal real subfield of the field of $2^{r+2}$th roots of unity. Thus $\mathbb{Q}_0 = \mathbb{Q}$ and $\mathbb{Q}_1 = \mathbb{Q}(\sqrt{2})$. Also let $\mathbb{Q}_\infty$ denote the union of the $\mathbb{Q}_r$ for all natural numbers $r$. This is the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$. The cyclotomic $\mathbb{Z}_2$-extension of a field $L$ is $L_\infty = L \cdot \mathbb{Q}_\infty$, and the $n$-th layer of this extension $L_n$ is the unique subfield of $L_\infty$ of degree $2^n$ over $L$.

**Proposition 2.3.** *The cohomology groups*

$$H^1(G, K_2(\mathcal{O}_E^S)) = K_2(\mathcal{O}_E^S)^-/K_2(\mathcal{O}_E^S)^{1-\tau}$$

*and*

$$H^2(G, K_2(\mathcal{O}_E^S)) = K_2(\mathcal{O}_E^S)^G/K_2(\mathcal{O}_E^S)^{1+\tau}$$

*both have order 1 if $E = F_1$ is the first layer in the cyclotomic $\mathbb{Z}_2$-extension of $F$, and both have order 2 otherwise.*

*Proof.* Since $K_2(\mathcal{O}_E^S)$ is finite and $G$ is cyclic, both cohomology groups have the same order (see [1, Prop.11]). To compute the order of $H^2(G, K_2(\mathcal{O}_E^S)) = K_2(\mathcal{O}_E^S)^G/K_2(\mathcal{O}_E^S)^{1+\tau}$, view it as the cokernel of the map $K_2(\mathcal{O}_E^S) \stackrel{1+\tau}{\to} K_2(\mathcal{O}_E^S)^G$. A standard functorial property states that this map factors as the transfer map $K_2(\mathcal{O}_E^S) \stackrel{\text{Trans}}{\to} K_2(\mathcal{O}_F^S)$ followed by the map $K_2(\mathcal{O}_F^S) \stackrel{\iota_*}{\to} K_2(\mathcal{O}_E^S)^G$ induced by the inclusion $\mathcal{O}_F^S \stackrel{\iota}{\to} \mathcal{O}_E^S$ of rings. Since the transfer is surjective, by Theorem, 2.1, we are reduced to computing the order of the cokernel of $\iota_*$. By Proposition 2.2, the domain and codomain of this map have the same order, and thus the order of the cokernel equals the order of the kernel. The order of $\ker(\iota_*)$ is computed in [11, Lemma 7.3], yielding 1 if $E = F_1$ and 2 otherwise. $\square$

## III. Cohomology of $W_2(E)$

We begin with two simple lemmas.

**Lemma 3.1.** *If $L$ is a real field, and $L \cap \mathbb{Q}_\infty = \mathbb{Q}_r$, then $2^{r+3}$ exactly divides $w_2(L)$. In particular, $w_2(L)$ is always divisible by 8.*

*Proof.* This follows from the fact that $L(\mu_{2^{r+2+k}})$ is the composite of the cyclic extensions $L(\mu_4)$ and $L \cdot \mathbb{Q}_{r+k}$. See [11, Lemma 7.2] for full details. $\square$

**Lemma 3.2.** *Define the positive integers $r$ and $s$ by $F \cap \mathbb{Q}_\infty = \mathbb{Q}_r$, and $E \cap \mathbb{Q}_\infty = \mathbb{Q}_s$. Then $s > r$ if and only if $E = F_1$, the first layer of the* *cyclotomic $\mathbb{Z}_2$-extension of $F$. (In fact, $s \le r+1$ always holds.)*

*Proof.* We have $s > r$ if and only if $E \supset \mathbb{Q}_{r+1} \not\subset F$ if and only if $E \supset F \cdot \mathbb{Q}_{r+1} = F_1 \supset F$. Since $[E : F] = 2 = [F_1 : F]$, this last condition is equivalent to $E = F_1$. $\square$

For any prime number $p$, let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ denote the localization of $\mathbb{Z}$ at the prime ideal $(p)$. If $M$ is a $\mathbb{Z}$-module, then similarly let $M_{(p)} \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ be the localization of $M$ at $(p)$. Note that if $M$ is a finite abelian group, we may identify $M_{(p)}$ with the Sylow $p$-subgroup of $M$, which we call the $p$-part of $M$. If $M$ is a $\mathbb{Z}[G]$-module, then $M_{(p)}$ becomes a $\mathbb{Z}_{(p)}[G]$-module. If $I$ is an ideal of $\mathcal{R} = \mathbb{Z}[G]$, we may identify $I_{(p)}$ with the extended ideal $I\mathbb{Z}_{(p)}[G] \subset \mathbb{Z}_{(p)}[G] = \mathcal{R}_{(p)} \subset \mathbb{Q}[G]$.

**Corollary 3.3.** *We have $W_2(E)_{(2)} = W_2(F)_{(2)}$ if and only if $E \ne F_1$.*

*Proof.* Since $W_2(E)$ is cyclic and contains $W_2(F)$, this follows upon applying the two preceeding lemmas. $\square$

**Proposition 3.4.** *The cohomology groups*

$$H^1(G, W_2(E)) = W_2(E)^-/W_2(E)^{1-\tau}$$

*and*

$$H^2(G, W_2(E)) = W_2(E)^G/W_2(E)^{1+\tau} = W_2(F)/W_2(E)^{1+\tau}$$

*have order 1 if $E = F_1$ is the first layer in the cyclotomic $\mathbb{Z}_2$ extension of $F$, and order 2 otherwise.*

*Proof.* Since $W_2(E)$ is finite and $G$ is cyclic, the two cohomology groups have the same order. We will compute $H^2(G, W_2(E))$. Since $|G| = 2$, the cohomology groups have exponent 2. Expressing $W_2(E)$ as a direct product (as a $\mathbb{Z}[G]$-module) of its 2-part $W_2(E)_{(2)}$ and its odd part, we therefore need only consider the cohomology of $W_2(E)_{(2)}$.

According to Lemma 3.1, we may write $w_2(F) = 8m$. The action of $\tau$ on the cyclic group $W_2(E)$ is non-trivial and fixes $W_2(F)$. Hence $\tau$ must act as $1+kw_2(F)$ for some integer $k$. Thus $W_2(E)_{(2)}^{1+\tau} = W_2(E)_{(2)}^{2+kw_2(F)} = (W_2(E)_{(2)}^{1+4m})^2 = W_2(E)_{(2)}^2$. Since $W_2(E)^G = W_2(F)$, we conclude that $H^2(G, W_2(E)_{(2)}) = W_2(F)_{(2)}/W_2(E)_{(2)}^2$. Again using the cyclicity of $W_2(E)$, we see that this cohomology group is cyclic of exponent 2 and is non-trivial if and only if $W_2(E)_{(2)} = W_2(F)_{(2)}$. By the corollary, this is equivalent to $E \neq F_1$. (Here we also see that $s \leq r + 1$.) Thus the cohomology groups are of order 2 when $E \neq F_1$, and are trivial otherwise. □

## IV. The Annihilator of $W_2(E)$

We have observed that $\tau$ acts on $W_2(E)$ as $N = 1 + kw_2(F)$, for some integer $k$. Thus $\tau - N$ and $|W_2(E)|$ lie in $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$. It is easy to see that these two elements generate $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$. For if $a\tau + b \in \mathrm{Ann}_{\mathcal{R}}(W_2(E))$, then $a\tau + b - a(\tau - N) = b + aN \in \mathrm{Ann}_{\mathcal{R}}(W_2(E))$. We see that $(b + aN) = c|W_2(E)|$, for some $c \in \mathbb{Z}$ and so $a\tau + b = a(\tau - N) + c|W_2(E)|$.

For future purposes, we would like to express $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$ in terms of $1 + \tau$ and $1 - \tau$.

**Proposition 4.1.** $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$ *is generated by* $|W_2(E)^{1+\tau}|(1+\tau)$ *and* $|W_2(E)^{1-\tau}|(1-\tau)$ *when* $E \neq F_1$. *It is generated by* $|W_2(E)^{1+\tau}|\frac{1+\tau}{2} + |W_2(E)^{1-\tau}|\frac{1-\tau}{2}$ *when* $E = F_1$.

*Proof.* Note that $1-\tau$ induces an isomorphism from $W_2(E)/W_2(F)$ to $W_2(E)^{1-\tau}$. Thus from Cor. 3.3, we have $E = F_1$ if and only if 2 divides $|W_2(E)/W_2(F)|$, which holds if and only if 2 divides $|W_2(E)^{1-\tau}|$. Also note that the greatest common divisor of the orders of the subgroups $W_2(E)^{1+\tau}$ and $W_2(E)^{1-\tau}$ of the cyclic group $W_2(E)$ is the order of their intersection. This intersection clearly has exponent 2 and is cyclic. From Lemma 3.1 and Proposition 3.4, we know that the first group $W_2(E)^{1+\tau}$ has order divisible by 4. We conclude that this greatest common divisor is 2 if $E = F_1$ and 1 otherwise.

The elements $\alpha^+ = |W_2(E)^{1+\tau}|(1+\tau)$ and $\alpha^- = |W_2(E)^{1-\tau}|(1-\tau)$ clearly annihilate $W_2(E)$. When $E \neq F_1$, we show that these generate $|W_2(E)|$ and $\tau - N$ for some $N$, which in turn generate $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$. In this case, the integers $|W_2(E)^{1+\tau}|$ and $|W_2(E)^{1-\tau}|$ are relatively prime, so an integer linear combination of $\alpha^+$ and $\alpha^-$ yields the desired element of the form $\tau - N$. Also $|W_2(E)|/2$ is a common multiple of $|W_2(E)^{1+\tau}|$ and $|W_2(E)^{1-\tau}|$, as $|W_2(E)|/\|WE^{1+\tau}| = |W_2(E)^-|$, where $-1 \in W_2(E)^-$; while $|W_2(E)|/|W_2(E)^{1-\tau}| = |W_2(E)^G| = |W_2(F)| = 8m$ by Lemma 3.1 again. So $\alpha^+$ and $\alpha^-$ generate $(|W_2(E)|/2)(1+\tau+1-\tau) = |W_2(E)|$.

Now suppose that $E = F_1$. We first show that $\gamma = |W_2(E)^{1+\tau}|\frac{1+\tau}{2} + |W_2(E)^{1-\tau}|\frac{1-\tau}{2} = a\frac{1+\tau}{2} + b\frac{1-\tau}{2} \in \mathrm{Ann}_{\mathcal{R}}(W_2(E))$. We have seen that the greatest common divisor of the two orders $a$ and $b$ here is 2. If $\omega =$

$\omega_1^2 \in W_2(E)^2$, then $\omega^\gamma = (\omega_1^{1+\tau})^{|W_2(E)^{1+\tau}|}(\omega_1^{1-\tau})^{|W_2(E)^{1-\tau}|} = 1$. On the other hand, if $\omega \notin W_2(E)^2$, then its image generates $W_2(E)/W_2(E)^2$, and thus the image of $\omega^{1+\tau}$ generates $W_2(E)^{1+\tau}/(W_2(E)^{1+\tau})^2$, and finally $\omega^{|W_2(E)^{1+\tau}|\frac{1+\tau}{2}}$ generates $W_2(E)^{(1+\tau)|W_2(E)^{1+\tau}|/2}$ which is cyclic of order 2 and therefore equals $\{-1\}$. Similarly $\omega^{|W_2(E)^{1-\tau}|\frac{1-\tau}{2}} = -1$. Thus $\omega^\gamma = -1 \cdot -1 = 1$ Next we show that $\gamma$ generates both $\tau - N$ for some $N$, and $|W_2(E)|$. Since $\gcd(a/2, b/2) = 1$ and $b/2$ is odd while $a/2$ is even, we have $\gcd(a, \frac{a}{2} - \frac{b}{2}) = 1$. This shows that $\tau - N$ for some $N$ is an integer linear combination of $(1+\tau)\gamma = a\tau + a$ and $\gamma = (\frac{a}{2} - \frac{b}{2})\tau + (\frac{a}{2} + \frac{b}{2})$. Also, $\frac{|W_2(E)|}{2}$ is a common multiple of $a$ and $b$, as seen above, and hence $|W_2(E)| = \frac{|W_2(E)|}{2}(1 + \tau + 1 - \tau)$ is generated by $\gamma(1+\tau) = a(1+\tau)$ and $\gamma(1-\tau) = b(1-\tau)$. $\square$

## V. Properties of Fitting Ideals

In the following Proposition, we list the standard properties of Fitting ideals which we will need. So suppose that $A$ is a commutative Noetherian ring and $M$ is a finitely generated $A$-module. We denote the Fitting ideal of $M$ over $A$ as $\mathrm{Fit}_A(M)$. It is the ideal of $A$ generated by the determinants of all square matrices representing relations among a set of generators of $M$.

**Proposition 5.1.**

1. *If $M$ is a cyclic $A$-module, then* $\mathrm{Fit}_A(M) = \mathrm{Ann}_A(M)$.
2. *Given a morphism of commutative Noetherian rings $f : A \to B$, we have* $\mathrm{Fit}_B(M \otimes_A B) = f(\mathrm{Fit}_A(M))B$. *In particular,*

*a. If* $A \subset B$*, then* $\mathrm{Fit}_B(M \otimes_A B) = \mathrm{Fit}_A(M)B$*.*

*b. If* $I$ *is an ideal of* $A$*, then* $\mathrm{Fit}_{A/I}(M/IM) = (\mathrm{Fit}_A(M) + I)/I$*.*

*c. If* $T$ *is a multiplicatively closed set in* $A$ *and* $A_T = T^{-1}A$ *(resp.* $T^{-1}M = M_T$*) is the corresponding ring (resp. module) of fractions, then* $\mathrm{Fit}_{A_T}(M_T) = (\mathrm{Fit}_A(M))A_T$*.*

*3. If* $A = \mathbb{Z}$ *and* $M$ *is finite, then* $\mathrm{Fit}_{\mathbb{Z}}(M) = |M|\mathbb{Z}$*.*

*4. If* $A = A_1 \oplus A_2$ *and correspondingly* $M = M_1 \oplus M_2$*, then* $\mathrm{Fit}_A(M) = \mathrm{Fit}_{A_1}(M_1) \oplus \mathrm{Fit}_{A_2}(M_2)$

*Proof.* Proofs of parts 1–3 may be found in [2]. The proof of part 4, and indeed every part, is a straightforward exercise. □

We will also use a more specialized result on Fitting ideals, variations of which are found in [4] and [12].

**Proposition 5.2.** *Suppose that* $G$ *is a finite abelian group and that* $M$ *is a finite* $\mathbb{Z}[G]$*-module. If* $M$ *is cohomologically trivial, then* $\mathrm{Fit}_{\mathbb{Z}_{(p)}[G]}(M_{(p)})$ *is principal, for any prime* $p$*.*

*Proof.* It is a standard fact [1, Thm. 9] that $M$ is cohomologically trivial if and only if its projective dimension as a $\mathbb{Z}[G]$-module is less than or equal to 1. This means that we have a resolution of $M$ by projective modules

$$0 \to Q \to P \to M \to 0,$$

and we may choose $P$ to be free of some rank $n$. Localization preserves exactness, so we get

$$0 \to Q_{(p)} \to P_{(p)} \to M_{(p)} \to 0.$$

Now $\mathbb{Z}_{(p)}[G]$ is a semilocal ring (its maximal ideals correspond to those of the finite ring $\mathbb{Z}/p\mathbb{Z}[G]$), while $P_{(p)}$ is a free $\mathbb{Z}_{(p)}[G]$-module of rank $n$ and $Q_{(p)}$ is a projective submodule. We claim that it has constant rank $n$. This is because if we localize the last exact sequence at a minimal prime (necessarily contained in the zero-divisors) of $\mathbb{Z}_{(p)}[G]$, the order of $M$ will be inverted and the middle term stays free of rank $n$, while the term to the right becomes 0 and hence the term to the left must be free of rank $n$. Hence if we localize the projective module $Q_{(p)}$ at any prime of $\mathbb{Z}_{(p)}[G]$, it must be free, and further localization at a minimal prime shows that the rank must be $n$. But a finitely generated projective module of constant rank $n$ over a semilocal ring is free of rank $n$ (see eg. [2, Exercise 4.13]). In this situation, the Fitting ideal $\mathrm{Fit}_{\mathbb{Z}_{(p)}[G]}(M_{(p)})$ is clearly generated by the determinant of the single $n$ by $n$ matrix representing the map in our exact sequence between the free module $Q_{(p)}$ and the free module $P_{(p)}$. $\square$

## VI. Fitting ideals in the maximal order of $\mathbb{Q}[G]$

We will have occasion to extend ideals from $\mathcal{R} = \mathbb{Z}[G] = \mathbb{Z} \oplus \mathbb{Z}\tau$ to $\mathcal{S} = \mathbb{Z}\frac{1+\tau}{2} \oplus \mathbb{Z}\frac{1-\tau}{2}$, the maximal order in $\mathbb{Q}[G]$. Note that $\mathcal{S} \cong \mathcal{R}/(1-\tau) \oplus \mathcal{R}/(1+\tau)$ under the obvious homomorphism, the inverse homomorphism being $(\overline{r_1}, \overline{r_2}) \to r_1\frac{1+\tau}{2} + r_2\frac{1-\tau}{2}$. Also, $\mathbb{Z} \cong \mathcal{R}/(1 \pm \tau)$ via the homomorphism sending each element of $\mathbb{Z}$ to its coset. We use these isomorphisms to make identifications in the next lemma.

**Lemma 6.1.** *Suppose that $M$ is a finite $\mathcal{R}$-module. Then the $\mathcal{S}$-ideal $\mathrm{Fit}_{\mathcal{R}}(M)\mathcal{S}$ is generated by $|M^G|\frac{1+\tau}{2} + |M^-|\frac{1-\tau}{2}$.*

*Proof.* Note that $|M^G| = |M/M^{1-\tau}|$ since $M/M^G \cong M^{1-\tau}$, and similarly $|M^-| = |M/M^{1+\tau}|$. Using part 2a of Proposition 5.1, then part 4, and then part 3, we get

$$\mathrm{Fit}_{\mathcal{R}}(M)\mathcal{S} = \mathrm{Fit}_{\mathcal{S}}(M \otimes_{\mathcal{R}} \mathcal{S}) = \mathrm{Fit}_{\mathcal{R}/(1-\tau)\oplus\mathcal{R}/(1+\tau)}(M/M^{1-\tau} \oplus M/M^{1+\tau})$$

$$= \mathrm{Fit}_{\mathcal{R}/(1-\tau)}(M/M^{1-\tau}) \oplus \mathrm{Fit}_{\mathcal{R}/(1+\tau)}(M/M^{1+\tau})$$

$$= (|M/M^{1-\tau}|\frac{1+\tau}{2} + |M/M^{1+\tau}|\frac{1-\tau}{2})\mathcal{S} = (|M^G|\frac{1+\tau}{2} + |M^-|\frac{1-\tau}{2})\mathcal{S} \quad \square$$

Let $k^+ = |K_2(\mathcal{O}_E^S)^G|$ (which equals $|K_2(\mathcal{O}_F^S)|$ by Proposition 2.2), and $k^- = |K_2(\mathcal{O}_E^S)^-|$. Similarly let $w^+ = |W_2(E)^G| = |W_2(F)|$ and $w^- = |W_2(E)^-|$.

**Proposition 6.2.**

$$\mathcal{I}^{\mathrm{Fit}}\mathcal{S} = \big(k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2}\big)\mathcal{S}$$

*and*

$$\mathrm{Ann}_{\mathcal{R}}(W_2(E))\mathcal{S} = \mathrm{Fit}_{\mathcal{R}}(W_2(E))\mathcal{S} = \big(w^+\frac{1+\tau}{2} + w^-\frac{1-\tau}{2}\big)\mathcal{S}$$

*Proof.* Apply the lemma to $K_2(\mathcal{O}_E^S)$ and $W_2(E)$, and use part 1 of Prop. 5.1. $\square$

Our goal of course is to compute $\mathcal{I}^{\mathrm{Fit}} = \mathrm{Fit}_{\mathcal{R}}(K_2(\mathcal{O}_E^S))$; we will also need $\mathrm{Ann}_{\mathcal{R}}(W_2(E)) = \mathrm{Fit}_{\mathcal{R}}(W_2(E))$. Prop. 6.2 will help us identify the former. The latter is known and easily done directly. We will derive it in a form most useful to us in the next section. That, combined with Proposition 6.2 and the evaluation of $\theta_{E/F}^S(-1)$ will be enough to provide the odd part of our main result (Theorem 1.3). To obtain the 2-part of Theorem 1.3, we will explicitly compute a key element of $\mathcal{I}^{\mathrm{Fit}}$ in Section IX.

## VII. The Stickelberger Ideal $\mathcal{I}^{\mathrm{Sti}} = \mathrm{Ann}_{\mathcal{R}}(W_2(E))\theta^S_{E/F}(-1)$

Associated with each character $\chi$ of $G$, we have the Artin $L$-function with Euler factors for primes in $S$ removed:

$$L^S_{E/F}(s,\chi) = \sum_{\substack{\mathfrak{a}\text{ integral}\\ (\mathfrak{a},S)=1}} \frac{\chi(\sigma_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}^s} = \prod_{\text{prime }\mathfrak{p}\notin S} \big(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathrm{N}\mathfrak{p}^s}\big)^{-1}.$$

These are related to the equivariant $L$-function by

$$\theta^S_{E/F}(s) = \sum_{\chi} L^S_{E/F}(s,\overline{\chi})e_{\chi},$$

where $\overline{\chi}$ is the complex conjugate of $\chi$, as the two sides of this equation agree after multiplication by each idempotent $e_\chi = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$. In our case we just have the trivial character $\chi_0$ and the non-trivial character $\chi_1$, giving

$$\begin{aligned}\theta^S_{E/F}(s) = L^S_{E/F}(s,\chi_0)\frac{1+\tau}{2} + L^S_{E/F}(s,\chi_1)\frac{1-\tau}{2}\\ = \zeta^S_F(s)\frac{1+\tau}{2} + \frac{\zeta^S_E(s)}{\zeta^S_F(s)}\frac{1-\tau}{2},\end{aligned}$$

by standard properties of Artin $L$-functions. Thus

$$\theta^S_{E/F}(-1) = \zeta^S_F(-1)\frac{1+\tau}{2} + \frac{\zeta^S_E(-1)}{\zeta^S_F(-1)}\frac{1-\tau}{2}.$$

Referring now to Conjecture 1.1 and Theorem 1.2, we clearly obtain the following.

**Proposition 7.1.** *Assuming the Birch-Tate conjecture for $E$ and for $F$, we have*

$$(-1)^{|S|}\theta^S_{E/F}(-1) = \frac{|K_2(\mathcal{O}^S_F)|}{w_2(F)}\frac{1+\tau}{2} + (-1)^{|S_E|}\frac{w_2(F)}{w_2(E)}\frac{|K_2(\mathcal{O}^S_E)|}{|K_2(\mathcal{O}^S_F)|}\frac{1-\tau}{2}$$

*Equality of the odd parts holds unconditionally.*

**Lemma 7.2.** *Assuming the Birch-Tate conjecture for $E$ and $F$, we have $w^+\frac{1+\tau}{2}\theta^S_{E/F}(-1) = \pm k^+\frac{1+\tau}{2}$ and $w^-\frac{1-\tau}{2}\theta^S_{E/F}(-1) = \pm k^-\frac{1-\tau}{2}$. Equality of odd parts holds unconditionally.*

*Proof.* Using Proposition 7.1, we see that

$$\begin{aligned}\pm w^+\frac{1+\tau}{2}\theta^S_{E/F}(-1) &= \big(w^+\frac{1+\tau}{2}\big)\big(\frac{k^+}{w^+}\frac{1+\tau}{2} \pm \frac{w^+}{w_2(E)}\frac{|K_2(\mathcal{O}^S_E)|}{k^+}\frac{1-\tau}{2}\big)\\ &= k^+\frac{1+\tau}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\pm w^-\frac{1-\tau}{2}\theta^S_{E/F}(-1) &= \big(w^-\frac{1-\tau}{2}\big)\big(\frac{k^+}{w^+}\frac{1+\tau}{2} \pm \frac{w^+}{w_2(E)}\frac{|K_2(\mathcal{O}^S_E)|}{k^+}\frac{1-\tau}{2}\big)\\ &= \pm\frac{w^+w^-}{w_2(E)}\frac{|K_2(\mathcal{O}^S_E)|}{k^+}\frac{1-\tau}{2}.\end{aligned}$$

Here $w_2(E)/w^- = |W_2(E)^{1+\tau}|$, so

$$\frac{w^+w^-}{w_2(E)} = |W_2(E)^G|/|W_2(E)^{1+\tau}| = |K_2(\mathcal{O}^S_E)^G|/|K_2(\mathcal{O}^S_E)^{1+\tau}|,$$

by Prop. 3.4 and Prop. 2.3. By Prop 2.2, we also have $|K_2(\mathcal{O}^S_E)^G| = |K_2(\mathcal{O}^S_F)| = k^+$. Making these substitutions above leads to the equality

$$\pm(w^-\frac{1-\tau}{2})\theta^S_{E/F}(-1) = \big(|K_2(\mathcal{O}^S_E)|/|K_2(\mathcal{O}^S_E)^{1+\tau}|\big)\frac{1-\tau}{2} = k^-\frac{1-\tau}{2},$$

as desired. This proof works for the odd parts without assuming the Birch-Tate conjecture. □

**Proposition 7.3.** *Assuming the Birch-Tate conjecture for $E$ and $F$, we have $\mathcal{I}^{\mathrm{Sti}}\mathcal{S} = \mathcal{I}^{\mathrm{Fit}}\mathcal{S}$. The equality $\mathcal{I}^{\mathrm{Sti}}\mathcal{S}[1/2] = \mathcal{I}^{\mathrm{Fit}}\mathcal{S}[1/2]$ holds unconditionally.*

*Proof.* From Prop. 6.2 we have that $\mathcal{I}^{\mathrm{Fit}}\mathcal{S}$ is generated by $k^+\frac{1+\tau}{2}+k^-\frac{1-\tau}{2}$ as an ideal in $\mathcal{S}$. Similarly $\mathcal{I}^{\mathrm{Sti}}\mathcal{S} = \mathrm{Ann}_{\mathcal{R}}(W_2(E))\theta^S_{E/F}(-1)$ is generated by $(w^+\frac{1+\tau}{2} + w^-\frac{1-\tau}{2})\theta^S_{E/F}(-1)$. Thus it suffices to show that $k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2}$ and $(w^+\frac{1+\tau}{2}+w^-\frac{1-\tau}{2})\theta^S_{E/F}(-1)$ are associates in $\mathcal{S}$. This is easliy accomplished upon multiplying by one of the units $\pm1$ or $\pm\tau$, and using Proposition 7.2. If we first extend ideals to $\mathcal{S}[1/2]$, the same proof works unconditionally. $\square$

**Corollary 7.4.** $\mathcal{I}^{\mathrm{Fit}}\mathcal{R}[1/2] = \mathcal{I}^{\mathrm{Sti}}\mathcal{R}[1/2]$*. Consequently* $\mathcal{I}^{\mathrm{Fit}}_{(p)} = \mathcal{I}^{\mathrm{Sti}}_{(p)}$ *for all odd primes $p$.*

*Proof.* Noting that $\mathcal{R}[1/2] = \mathcal{S}[1/2]$, we see that the first equality is just a restatement of the unconditional part of Proposition 7.3. The second equality follows upon extending ideals from $\mathcal{R}[1/2] = \mathbb{Z}[1/2][G]$ to $\mathcal{R}_{(p)} = \mathbb{Z}_{(p)}[G]$. $\square$

Cor. 7.4 establishes the unconditional part of Theorem 1.3. Our goal now is to show that $\mathcal{I}^{\mathrm{Fit}}_{(2)} = \mathcal{I}^{\mathrm{Sti}}_{(2)}$, assuming the 2-part of the Birch-Tate conjecture.

## VIII. Ideals in $\mathcal{R}_{(2)} = \mathbb{Z}_{(2)}[G]$

Since $G$ is a 2-group, $\mathcal{R}_{(2)} = \mathbb{Z}_{(2)}[G]$ is a local ring. This is easily seen as it is an integral extension of the discrete valuation ring $\mathbb{Z}_{(2)}$, so any maximal ideal must contain 2, and hence must also contain $1-\tau$, whose square $2(1-\tau)$ is in the ideal generated by 2. The ideal $\mathfrak{m} = (2, 1-\tau) = (1+\tau, 1-\tau)$ is of index 2, hence is the unique maximal ideal.

We will need to consider the relationship between $\mathcal{R}_{(2)}$ and the over-ring $\mathcal{S}_{(2)} = \mathbb{Z}_{(2)}\frac{1+\tau}{2} \oplus \mathbb{Z}_{(2)}\frac{1-\tau}{2} \cong \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$, which is clearly a principal ideal ring.

**Lemma 8.1.** *The group of units $\mathcal{S}_{(2)}^\times$ of $\mathcal{S}_{(2)}$ equals the group of units of $\mathcal{R}_{(2)}$.*

*Proof.*

$$\mathcal{S}_{(2)}^\times = \mathbb{Z}_{(2)}^\times \frac{1+\tau}{2} \oplus \mathbb{Z}_{(2)}^\times \frac{1-\tau}{2} = (1+2\mathbb{Z}_{(2)})\frac{1+\tau}{2} \oplus (1+2\mathbb{Z}_{(2)})\frac{1-\tau}{2} = (\frac{1+\tau}{2} + \frac{1-\tau}{2}) + \mathbb{Z}_{(2)}(1+\tau) + \mathbb{Z}_{(2)}(1-\tau)$$
$$= 1 + \mathfrak{m} = \mathcal{R}_{(2)}^\times \quad \square$$

**Lemma 8.2.** *Suppose that $I$ is an ideal of finite index in $\mathcal{R}_{(2)}$. Let $\alpha$ be a generator in the principal ideal ring $\mathcal{S}_{(2)}$ for the extended ideal $I\mathcal{S}_{(2)}$. Then $\alpha \in I$.*

*Proof.* Modifying by units, we may assume that $\alpha = 2^i\frac{1+\tau}{2} + 2^j\frac{1-\tau}{2}$. Since $\mathcal{S} = \mathcal{R}\frac{1+\tau}{2} \oplus \mathcal{R}\frac{1-\tau}{2}$, we have that $2^i\frac{1+\tau}{2} + 2^j\frac{1-\tau}{2}$ generates $I\mathcal{S} = I\frac{1+\tau}{2} \oplus I\frac{1-\tau}{2}$. Thus $2^i\frac{1+\tau}{2}$ generates $I\frac{1+\tau}{2}$ over $\mathbb{Z}_{(2)}$ and $2^j\frac{1-\tau}{2}$ generates $I\frac{1-\tau}{2}$. Consequently $2^i\frac{1+\tau}{2} + 2^{j'}\frac{1-\tau}{2} \in I$ for some $j' \geq j$ and $2^{i'}\frac{1+\tau}{2} + 2^j\frac{1-\tau}{2} \in I$ for some $i' \geq i$. Multiplying the second element by $(1-\tau) \in \mathcal{R}$, we also find that $2^{j+1}\frac{1-\tau}{2} \in I$. If $j' = j$, then we have $2^i\frac{1+\tau}{2} + 2^j\frac{1-\tau}{2} \in I$ as desired. If not, then $2^i\frac{1+\tau}{2} + 2^{j'}\frac{1-\tau}{2} \in I$ and $2^{j+1}\frac{1-\tau}{2} \in I$ generate

$2^i \frac{1+\tau}{2} \in I$. Combined with $2^{i'} \frac{1+\tau}{2} + 2^j \frac{1-\tau}{2} \in I$, this generates $2^j \frac{1-\tau}{2} \in I$, and finally $2^i \frac{1+\tau}{2} + 2^j \frac{1-\tau}{2} \in I$. □

**Remark 8.3** In the setting of Lemma 8.2, one can in fact show that $I$ is either the principal ideal generated by $\alpha$ or the non-prinicipal ideal generated by $\alpha\frac{1+\tau}{2}$ and $\alpha\frac{1-\tau}{2}$.

## IX. Computation of $\mathcal{I}_{(2)}^{\mathrm{Fit}}$

Assuming the 2-part of the Birch-Tate conjecture for $E$ and $F$, we now perform the computation of $\mathcal{I}_{(2)}^{\mathrm{Fit}}$, separating it into two different cases.

**Proposition 9.1.** *Suppose that $E = F_1$, and assume that the 2-part of the Birch-Tate conjecture holds for $F$ and for $E$. Then $\mathcal{I}_{(2)}^{\mathrm{Fit}} = \mathcal{I}_{(2)}^{\mathrm{Sti}}$.*

*Proof.* Since $E = F_1$, Proposition 2.3 and Proposition 5.2 show that $\mathcal{I}_{(2)}^{\mathrm{Fit}}$ is a principal ideal in $\mathcal{R}_{(2)}$. Let $\alpha \in \mathcal{R}_{(2)}$ be a generator for $\mathcal{I}_{(2)}^{\mathrm{Fit}}$. Similarly, Proposition 3.4 and Proposition 5.2 show that $\mathrm{Ann}_R(W_2(E))\mathcal{R}_{(2)}$ and hence $\mathcal{I}_{(2)}^{\mathrm{Sti}} = \theta_{E/F}^S(-1)\mathrm{Ann}_R(W_2(E))\mathcal{R}_{(2)}$ is principal. We can also see this more explicitly from Prop. 4.1. Let $\beta \in \mathcal{R}_{(2)}$ be a generator for $\mathcal{I}_{(2)}^{\mathrm{Sti}}$. From Proposition 7.3, we deduce that $\mathcal{I}_{(2)}^{\mathrm{Fit}}$ and $\mathcal{I}_{(2)}^{\mathrm{Sti}}$ extend to the same ideal in $\mathcal{S}_{(2)}$. Thus $\alpha$ and $\beta$ are associates in $\mathcal{S}_{(2)}$. By Lemma 8.1, they are associates in $\mathcal{R}_{(2)}$, and thus $\mathcal{I}_{(2)}^{\mathrm{Fit}} = \mathcal{I}_{(2)}^{\mathrm{Sti}}$. □

We turn now to the case of $E \neq F_1$, in which $|K_2(\mathcal{O}_E^S)^- / K_2(\mathcal{O}_E^S)^{1-\tau}| = 2$, by Prop. 2.3. We will use a lemma to simplify the computation.

**Lemma 9.2.** *Suppose that $M$ is a finite abelian 2-group, and $m \in M$ is an element of order 2. Then $M$ is isomorphic to a direct product of* *non-trivial cyclic groups, one of which contains $m$.*

*Proof.* We may assume that $M = \oplus_{i=1}^{r} \mathbb{Z}/2^{c_i}\mathbb{Z}$,with $1 \leq c_i \leq c_{i+1}$ for each $i$ between 1 and $r-1$, inclusive. We denote the standard generators by $e_1 = (\overline{1}, \overline{0}, \ldots, \overline{0}), \cdots, e_r = (\overline{0}, \ldots, \overline{0}, \overline{1})$. Then necessarily $m = (2^{c_1 - 1}\varepsilon_1, 2^{c_2 - 1}\varepsilon_2, \ldots)$, with $\varepsilon_i = \overline{1}$ or $\overline{0}$, and not all $\varepsilon_i = \overline{0}$. Let $t$ be the smallest index for which $\varepsilon_t \neq \overline{0}$. Put $e'_t = (\overline{0}, \cdots, \overline{0}, \varepsilon_t = \overline{1}, 2^{c_{t+1} - c_t}\varepsilon_{t+1}, \ldots, 2^{c_r - c_t}\varepsilon_r)$, so that $2^{c_t - 1}e'_t = m$. Then we can see that $M$ is the direct product of the subgroups generated by the elements $e_1, \ldots, e_{t-1}, e'_t, e_{t+1}, \ldots, e_r$; and this gives the desired decomposition of $M$. $\square$

**Proposition 9.3.** *If $E \neq F_1$, then $k^+ \frac{1+\tau}{2} \in \mathcal{I}_{(2)}^{\mathrm{Fit}}$.*

*Proof.* First, $\mathcal{I}_{(2)}^{\mathrm{Fit}} = \mathrm{Fit}_{\mathcal{R}_{(2)}}(K_2(\mathcal{O}_E^S)_{(2)})$, by part 2c) of Prop. 5.1. We know by Proposition 2.3 that $K_2(\mathcal{O}_E^S)^- / K_2(\mathcal{O}_E^S)^{1-\tau}$ has order 2, and it follows by decomposing $K_2(\mathcal{O}_E^S)$ into a 2-part and an odd part that $K_2(\mathcal{O}_E^S)^-_{(2)} / K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}$ has order 2. Now apply Lemma 9.2 with $M{=}K_2(\mathcal{O}_E^S)_{(2)} / K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}$ and $m$ as a generator of the subgroup $K_2(\mathcal{O}_E^S)^-_{(2)} / K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}$ of order 2. Let $\overline{\gamma_i}$, $1 \leq i \leq r$ be the generators of $K_2(\mathcal{O}_E^S)_{(2)} / K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}$ obtained from the lemma, with each $\gamma_i \in K_2(\mathcal{O}_E^S)_{(2)}$. Then $\overline{\gamma_i}$ has order $d_i = 2^{c_i}$, a positive power of 2 for each $i$, and $\prod_i d_i = |K_2(\mathcal{O}_E^S)_{(2)} / K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}|$. Denote this order by $k^+_{(2)}$. Again the decomposition of $K_2(\mathcal{O}_E^S)$ into an odd part and a 2-part shows that $k^+_{(2)}$ is the 2-part of $|K_2(\mathcal{O}_E^S)/K_2(\mathcal{O}_E^S)^{1-\tau}|$, which equals $|K_2(\mathcal{O}_F^S)| = |K_2(\mathcal{O}_E^S)^G| = k^+$ by Thm. 2.1 and Prop. 2.2. Thus $k^+$ and $k^+_{(2)}$ are associates in $\mathbb{Z}_{(2)} \subset \mathcal{R}_{(2)}$ and it suffices to show that $k^+_{(2)} \frac{1+\tau}{2} \in$

$\mathrm{Fit}_{\mathcal{R}_{(2)}}(K_2(\mathcal{O}_E^S)_{(2)})$.

The elements $\overline{\gamma_i}$, for $i = 1, \ldots, r$ generate $K_2(\mathcal{O}_E^S)_{(2)}/K_2(\mathcal{O}_E^S)_{(2)}^{1-\tau}$ as a group, hence as an $\mathcal{R}_{(2)}$-module. But $1 - \tau \in \mathfrak{m}$, the maximal ideal of the local ring $\mathcal{R}_{(2)}$, and hence by Nakayama's lemma, the elements $\gamma_i$ for $i = 1, \ldots, r$ generate $K_2(\mathcal{O}_E^S)_{(2)}$ as an $\mathcal{R}_{(2)}$-module. Finding $r$ relations among these $r$ generators will provide an element of the Fitting ideal in question by taking the determinant of the corresponding $r$-by-$r$ relations matrix.

The fact that the elements $\gamma_i$, for $i = 1, \ldots, r$ generate $K_2(\mathcal{O}_E^S)_{(2)}$ over $\mathcal{R}_{(2)}$ implies that the elements $\gamma_i^{1-\tau}$ for $i = 1, \ldots, r$ generate $K_2(\mathcal{O}_E^S)_{(2)}^{1-\tau}$. For each $i$, $\overline{\gamma_i}$ has order $d_i$, and this means that $\gamma_i^{d_i} \in K_2(\mathcal{O}_E^S)_{(2)}^{1-\tau}$. Multiplying by $\gamma_i^{(\tau-1)d_i/2} \in K_2(\mathcal{O}_E^S)_{(2)}^{1-\tau}$ gives us $\gamma_i^{(1+\tau)d_i/2} \in K_2(\mathcal{O}_E^S)_{(2)}^{1-\tau}$ for each $i$. As the elements $\gamma_i^{1-\tau}$ for $i = 1, \ldots, r$ generate this last $\mathcal{R}_{(2)}$-module, we have that, for each $i$,

$$\gamma_i^{(1+\tau)d_i/2} = \prod_{j=1}^{r} (\gamma_j^{1-\tau})^{b_{ij}},$$

for some elements $b_{ij} \in \mathcal{R}_{(2)}$. Let $D$ be the $r$ by $r$ diagonal matrix of the $d_i$, and $B$ be the matrix of the $b_{ij}$. Then the last equation shows that $\frac{1+\tau}{2}D - (1-\tau)B$ is a relations matrix for the generators $\gamma_i$ of $K_2(\mathcal{O}_E^S)_{(2)}$. Hence

$$\delta = \det\big(\frac{1+\tau}{2}D - (1-\tau)B\big) \in \mathrm{Fit}_{\mathcal{R}_{(2)}}(K_2(\mathcal{O}_E^S)_{(2)}) = \mathcal{I}_{(2)}^{\mathrm{Fit}}.$$

Modulo $(1-\tau)$, we have $\delta \equiv \det(\frac{1+\tau}{2}D) = (\frac{1+\tau}{2})^r \det(D) = \frac{1+\tau}{2}\prod_i d_i = \frac{1+\tau}{2}k_{(2)}^+$. Before considering $\delta$ modulo $1+\tau$, recall that the application of Lemma 9.2 also gives us that $\overline{\gamma_t}^{d_t/2}$ generates $K_2(\mathcal{O}_E^S)^-_{(2)}/K_2(\mathcal{O}_E^S)^{1-\tau}_{(2)}$. It follows that $\gamma_t^{d_t/2} \in K_2(\mathcal{O}_E^S)^-_{(2)}$, and consequently $\gamma_t^{(d_t/2)(1+\tau)}$ is trivial. Hence we may choose $b_{tj} = 0$ for all $j$, making $\det(B) = 0$. Now we see that $\delta \equiv \det(-2B) = 0$ modulo $(1+\tau)$. Thus $\delta - k^+_{(2)}\frac{1+\tau}{2} \in (1+\tau)\cap(1-\tau) = (0)$ in $\mathcal{R}_{(2)}$. We conclude that $k^+_{(2)}\frac{1+\tau}{2} = \delta \in \mathcal{I}^{\mathrm{Fit}}_{(2)}$, and we know that this implies the desired result. $\square$

**Proposition 9.4.** *Suppose that $E \neq F_1$, and assume that the Birch-Tate conjecture holds for $F$ and $E$. Then $\mathcal{I}^{\mathrm{Fit}}_{(2)} = \mathcal{I}^{\mathrm{Sti}}_{(2)}$.*

*Proof.* By Proposition 6.2, $k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2}$ generates $\mathcal{I}^{\mathrm{Fit}}\mathcal{S}$ as an ideal of $\mathcal{S}$, and hence this same element generates the extended ideal $\mathcal{I}^{\mathrm{Fit}}\mathcal{S}_{(2)}$ as an ideal of $\mathcal{S}_{(2)}$. Proposition 8.2 then implies that $k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2} \in \mathcal{I}^{\mathrm{Fit}}_{(2)}$. On the other hand, Proposition 9.3 gives $k^+\frac{1+\tau}{2} \in \mathcal{I}^{\mathrm{Fit}}_{(2)}$. Subtracting, we see that both $k^+\frac{1+\tau}{2}$ and $k^-\frac{1-\tau}{2}$ lie in $\mathcal{I}^{\mathrm{Fit}}_{(2)}$. Thus

$$\begin{aligned}\mathcal{I}^{\mathrm{Fit}}_{(2)} &\supset k^+\frac{1+\tau}{2}\mathcal{R}_{(2)} + k^-\frac{1-\tau}{2}\mathcal{R}_{(2)} \\ &= (k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2})(\mathcal{R}_{(2)}\frac{1+\tau}{2} + \mathcal{R}_{(2)}\frac{1-\tau}{2}) \\ &= (k^+\frac{1+\tau}{2} + k^-\frac{1-\tau}{2})\mathcal{S}_{(2)} = \mathcal{I}^{\mathrm{Fit}}_{(2)}\mathcal{S}_{(2)} \supset \mathcal{I}^{\mathrm{Fit}}_{(2)},\end{aligned}$$

by Prop. 6.2 again. We conclude that $\mathcal{I}^{\mathrm{Fit}}_{(2)} = k^+\frac{1+\tau}{2}\mathcal{R}_{(2)} + k^-\frac{1-\tau}{2}\mathcal{R}_{(2)}$. Turning to $\mathcal{I}^{\mathrm{Sti}}_{(2)}$, we have from Proposition 4.1 that $\mathrm{Ann}_{\mathcal{R}}(W_2(E))$ is generated by $|W_2(E)^{1+\tau}|(1+\tau)$ and $|W_2(E)^{1-\tau}|(1-\tau)$. By Proposition 2.3, these generators may also be written as $w^+\frac{1+\tau}{2}$ and $w^-\frac{1-\tau}{2}$. Hence $\mathcal{I}^{\mathrm{Sti}} = \theta^S_{E/F}(-1)\mathrm{Ann}_{\mathcal{R}}(W_2(E))$ is generated by $w^+\frac{1+\tau}{2}\theta^S_{E/F}(-1)$ and $w^-\frac{1-\tau}{2}\theta^S_{E/F}(-1)$. Using Lemma 7.2, we see that these generators equal $\pm k^+\frac{1+\tau}{2}$ and $\pm k^-\frac{1-\tau}{2}$. Consequently $\mathcal{I}^{\mathrm{Sti}}$ is generated by $k^+\frac{1+\tau}{2}$

and $k^{-}\frac{1-\tau}{2}$, so

$$\mathcal{I}^{\mathrm{Sti}}_{(2)} = k^{+}\frac{1+\tau}{2}\mathcal{R}_{(2)} + k^{-}\frac{1-\tau}{2}\mathcal{R}_{(2)} = \mathcal{I}^{\mathrm{Fit}}_{(2)}. \quad \square$$

Now we can complete the proof of Theorem 1.3. Cor. 7.4 proves the unconditional statement in Theorem 1.3, and shows that

$$\mathcal{I}^{\mathrm{Fit}}_{(p)} = \mathcal{I}^{\mathrm{Sti}}_{(p)} \subset \mathbb{Z}_{(p)}[G] \subset \mathbb{Q}[G],$$

for each odd prime $p$. Assuming the Birch-Tate conjecture for $E$ and $F$, Prop. 9.1 and Prop. 9.4 show that $\mathcal{I}^{\mathrm{Fit}}_{(2)} = \mathcal{I}^{\mathrm{Sti}}_{(2)}$. Note that for any ideal $I$ of $\mathcal{R}$, the intersection over all rational primes $\cap_p I_{(p)} = I$, because for each $\alpha \in \cap_p \mathcal{I}^{\mathrm{Fit}}_{(p)}$, the $\mathbb{Z}$-ideal $\{a \in \mathbb{Z} : a\alpha \in I\}$ is contained in no prime ideal of $\mathbb{Z}$, hence equals $\mathbb{Z}$. Thus

$$\mathcal{I}^{\mathrm{Fit}} = \cap_p \mathcal{I}^{\mathrm{Fit}}_{(p)} = \cap_p \mathcal{I}^{\mathrm{Sti}}_{(p)} = \mathcal{I}^{\mathrm{Sti}}. \quad \square$$

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