

SOME GALOIS MODULE
STRUCTURE ANALOGIES
BETWEEN TAME KERNELS
AND IDEAL CLASS GROUPS.

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ABSTRACT. Brumer's conjecture states that Stickelberger elements combining values of L-functions at $s=0$ for an abelian extension of number fields E/F should annihilate the ideal class group of E , when it is considered as a module over the appropriate group ring. We describe some cases in which an even stronger relationship is known to hold: an ideal obtained from these Stickelberger elements equals a Fitting ideal connected with the ideal class group. We consider the analog of this at $s=-1$, where the class group is replaced by the tame kernel, which we will define. Our recent work shows that for a field extension of degree 2, there is an exact equality between the Fitting ideal of an enlarged tame kernel and the most natural higher Stickelberger ideal; the 2-part of this equality is conditional on the Birch-Tate conjecture. Extensions of this result will be described as time permits.

I. THE CAST OF CHARACTERS

- E/F = an abelian Galois extension of algebraic number fields.

- $G = \text{Gal}(E/F)$, or order $|G|$.

- S = a finite set of primes of F which contains all of the infinite primes of F and all of the primes which ramify in E .

- S_E = the set of primes of E above those in S

- \mathcal{O}_E^S = the ring of S_E -integers of E .

- Cl_E^S = the S_E -class group of E

(\cong the ideal class group of \mathcal{O}_E^S), a $\mathbb{Z}[G]$ -module.

Note that Cl_E^S surjects onto $\text{Cl}_E^{S'}$ when $S \subset S'$.

- μ_E = the group of roots of unity of E , a $\mathbb{Z}[G]$ -module.

- \mathfrak{p} runs through the primes of F not in S
- \mathfrak{a} runs through the integral ideals of F which are relatively prime to each of the elements of S .
- $N\mathfrak{a} =$ the absolute norm of the ideal \mathfrak{a}
- $\sigma_{\mathfrak{a}} \in G$ is the well-defined automorphism attached to \mathfrak{a} via the Artin map.
- $e_{\chi} = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1}$, the idempotent in the group ring $\mathbb{C}[G]$ associated with a character χ of G .
- $\theta_{E/F}^S(s) = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a}, S)=1}} \frac{1}{N\mathfrak{a}^s} \sigma_{\mathfrak{a}}^{-1}$
 $= \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{N\mathfrak{p}^s} \sigma_{\mathfrak{p}}^{-1}\right)^{-1} = \sum_{\chi} L_{E/F}^S(s, \bar{\chi}) e_{\chi}$, the equivariant L -function with values in $\mathbb{C}[G]$. These series converge for the real part of s greater than 1 and the function they define extends meromorphically to \mathbb{C} .

II. THE S-CLASS GROUP AS A $\mathbb{Z}[G]$ -MODULE

For non-trivial results here, assume F totally real and E totally complex.

By Deligne-Ribet,

- $\mathbb{Z}[G]$ contains $\text{Stick}_{E/F}^S(0) = \text{Ann}_{\mathbb{Z}[G]}(\mu_E) \cdot \theta_{E/F}^S(0)$, the Stickelberger ideal.

By Stickelberger's theorem, when $F = \mathbb{Q}$,

- $\text{Stick}_{E/\mathbb{Q}}^S(0) \subset \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_E) \subset \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_E^S)$.

More generally, the Brumer-Stark conjecture (true for E abelian over \mathbb{Q} and most multi-quadratic extensions) implies

- $\text{Stick}_{E/F}^S(0) \subset \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_E) \subset \text{Ann}_{\mathbb{Z}[G]}(\text{Cl}_E^S)$.

QUESTION:

Is there a more precise relationship between Cl_E^S and $\text{Stick}_{E/F}^S(0)$, or some modified versions of these? (After all, the analytic class number formula gives more than just an annihilator of Cl_E !)

ANSWERS:

Usually consider minimal S , and fatten up $\text{Stick}_{E/F}^S(0)$ to $\text{Stick}'_{E/F}(0)$ by incorporating contributions from intermediate fields.

Taking minus parts (on which complex conjugation acts by inversion), Sinnott showed for $F = \mathbb{Q}$

- $|\mathbb{Z}[G]^- / \text{Stick}'_{E/\mathbb{Q}}(0)^-| = 2^a |\text{Cl}_E^-|$, with $a \in \mathbb{Z}^+$ specified.

Fitting ideals provide some other answers, generalizing the concept of order of a \mathbb{Z} -module to a finitely generated module M over a noetherian commutative ring R .

- $\text{Fitt}_R(M)$ is the ideal of R generated by the determinants of square matrices of relation vectors for a set of R -generators of M .
- $\text{Fitt}_R(M) \subset \text{Ann}_R(M)$, seen by multiplying a relations matrix by its adjoint.
- $\text{Fitt}_R(M) = \text{Ann}_R(M)$ for a cyclic R -module M , eg. μ_E .
- If R is local and M has projective dimension ≤ 1 , and is annihilated by a non-zero divisor, then $\text{Fitt}_R(M)$ is principal.

Iwasawa theory, and particularly Wiles' proof of the Main conjecture go into proving the next results.

MORE ANSWERS: Assume E is CM for these.

- In “nice extensions” of Greither, and cases when $F = \mathbb{Q}$ of Kurihara, $\text{Stick}'_{E/F}(0)^-$ and $\text{Fitt}_{\mathbb{Z}[G]}(\text{Cl}_E)^-$ agree at odd primes p .

- Assuming that the Equivariant Tamagawa Number Conjecture (ETNC) of Burns-Flach holds (eg. by Burns-Greither for E abelian over \mathbb{Q}), and a minor condition on μ_E , Greither (2006) obtains equality of odd minus-parts between the Fitting ideal of the *Pontrjagin dual* of Cl_E and $\text{Stick}'_{E/F}(0)$.

- For odd primes p , D. Solomon has defined an ideal in $\mathbb{Z}_p[G]$ via values of p -adic L -functions at $s = 1$ and a p -adic regulator. Assuming ETNC, Andrew Jones has shown that this ideal is contained in the extension of $\text{Fitt}_{\mathbb{Z}[G]}(\text{Cl}_E)$ to $\mathbb{Z}_p[G]$.

III. THE S -TAME KERNEL AS A $\mathbb{Z}[G]$ -MODULE

Assume E is totally real. Note that

- $\text{Cl}_E^S = \text{torsion subgroup of } K_0(\mathcal{O}_E^S)$.

Replace this by the S -tame kernel, known to be finite.

- $K_2(\mathcal{O}_E^S) \subset K_2(E) = (E^\times \otimes_{\mathbb{Z}} E^\times) / \langle \{x \otimes (1 - x)\} \rangle$

(Matsumoto's theorem)

- $K_2(\mathcal{O}_E^S)$ is the intersection of the kernels of all tame symbols for primes $\mathfrak{p} \notin S_E$: $\{\alpha, \beta\} = \overline{\alpha \otimes \beta} \mapsto (\alpha, \beta)_{\mathfrak{p}} = (-1)^{\nu_{\mathfrak{p}}(\alpha)\nu_{\mathfrak{p}}(\beta)} \alpha^{\nu_{\mathfrak{p}}(\beta)} / \beta^{\nu_{\mathfrak{p}}(\alpha)} \pmod{\mathfrak{p}} \in (\mathcal{O}_E/\mathfrak{p})^\times$.

Note that $K_2(\mathcal{O}_E^S)$ injects into $K_2(\mathcal{O}_E^{S'})$ when $S \subset S'$.

Replace μ_E by

- $W_2(E) = \text{the group of all roots of unity fixed by } \text{Gal}(E \cdot \mathbb{Q}^{ab}/E)^2$, a $\mathbb{Z}[G]$ -module on which an element of G acts by lifting and squaring.

The analog of the analytic class number formula is the Birch-Tate Conjecture:

- $|\zeta_F^S(-1)| = |K_2(\mathcal{O}_F^S)|/|W_2(F)|$.

The odd part is a consequence of Wiles' proof of the Main Conjecture, and the full conjecture holds for F abelian over \mathbb{Q} by Mazur-Wiles.

The higher Stickelberger ideal is

- $\text{Stick}_{E/F}^S(1) = \text{Ann}_{\mathbb{Z}[G]}(W_2(E))\theta_{E/F}^S(-1)$, again integral by Deligne-Ribet.

By the Coates-Sinott Theorem, when $F = \mathbb{Q}$

- $\text{Stick}_{E/F}^S(1)$ annihilates the odd part of $K_2(\mathcal{O}_E)$.

Generalized Coates-Sinnott Conjecture:

- $\text{Stick}_{E/F}^S(1) \subset \text{Ann}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_E^S)) \subset \text{Ann}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_E))$.

For multi-quadratic extensions, Sands-Simons show this holds, assuming the Birch-Tate conjecture.

QUESTION:

Is there a more precise relationship between $K_2(\mathcal{O}_E^S)$ and $\text{Stick}_{E/F}^S(1)$, or some modified versions of these? (After all, the Birch-Tate conjecture gives more than just an annihilator of the tame kernel!)

ANSWERS:

For relative quadratic and bi-quadratic extensions E/F satisfying the Birch-Tate conjecture:

- $|\mathbb{Z}[G]/\text{Stick}_{E/F}^S(1)| = |K_2(\mathcal{O}_E^S)|$

For relative quadratic extensions E/F satisfying the Birch-Tate conjecture:

- $\text{Stick}_{E/F}^S(1) = \text{Fitt}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_E^S)).$

IDEA OF PROOF: The Stickelberger ideal in a relative quadratic extension is computable by applying Birch-Tate twice. The maximal order in $\mathbb{Q}[G]$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z}[G]$ surjects onto both summands.

We extend ideals to this ring, where properties of Fitting ideals make them easy to compute. The equality of the images of the Fitting ideal and the Stickelberger ideal here follows from a computation which shows that $W_2(E)$ and $K_2(\mathcal{O}_E^S)$ have the same cohomology. This implies the odd part of the result. So consider the image of the Fitting ideal and the Stickelberger ideal in $\mathbb{Z}_2[G]$, a local ring. Knowing the extension to the maximal order in $\mathbb{Q}_2[G]$ leaves only two possibilities for each of these ideals, one principal and one non-principal. The Stickelberger ideal is computable: it is principal if and only if $W_2(E)$ is cohomologically trivial. The Fitting ideal must be principal when $K_2(\mathcal{O}_E^S)$ is cohomologically trivial. In the remaining case, a determinant argument using Nakayama's lemma shows that the Fitting ideal is not principal. Hence the two ideals agree since the two cohomology groups agree.

IV. EXTENSIONS

Other negative integers:

- Define $\text{Stick}_{E/F}(n) = \text{Ann}_{\mathbb{Z}[G]}(W_{n+1}(E))\theta_{E/F}^S(-n)$

- Full Generalized Coates-Sinnott Conjecture:

$\text{Stick}_{E/F}^S(n) \subset \text{Ann}_{\mathbb{Z}[G]}(K_{2n}(\mathcal{O}_E))$, any $n \in \mathbb{Z}^+$.

Many results related to this conjecture involve certain étale cohomology groups in place of algebraic K -groups. The Quillen-Lichtenbaum conjecture states that the two should be isomorphic. The unpublished Voevodsky-Rost Theorem proving this conjecture is becoming generally accepted. Given this, we have the following results.

- For almost all primes p , Burns-Greither (2005) actually prove equality of images in $\mathbb{Z}_p[G]$ of $\text{Stick}_{E/F}(n)$ and the plus or minus part of $\text{Fitt}_{\mathbb{Z}[G]}(K_{2n}(\mathcal{O}_E^S))$, assuming vanishing of an Iwasawa-theoretic p -adic μ -invariant, and Quillen-Lichtenbaum.

- For any odd prime p , James Barrett uses values of p -adic L -functions at $s = n + 1 \in \mathbb{Z}^+$ to define a higher analog of Solomon's ideal. Assuming ETNC and Quillen-Lichtenbaum, he shows that this ideal is contained in the extension of $\text{Fitt}_{\mathbb{Z}[G]}(K_{2n})$ to $\mathbb{Z}_p[G]$.

Predictions in the case of trivial zeroes:

In some situations, the Generalized Coates-Sinnott conjecture only predicts that 0 annihilates $K_{2n}(\mathcal{O}_E)$. Using derivatives of L -functions and assuming that a higher Stark conjecture holds so that rational values may be obtained from these in combinations with higher regulators, Snaitth proposes non-trivial annihilators. He proves his conjecture for $F = \mathbb{Q}$.

OPEN QUESTIONS:

Precisely how does the analogy work in connection with:

- Exceptional primes
- $p = 2$
- Changing the set S .
- Dualization.
- Contributions from intermediate fields.
- Reformulation using $\theta_{E/F}^S(1+n)$ a la Solomon, perhaps obviating dualization?