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Popescu's Conjecture in Multi-Quadratic Extensions

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Abstract. Popescu's conjecture connects special units in an abelian extension of a global field with the higher derivatives at 0 of L-functions attached to this extension. We prove the conjecture for certain multiquadratic extensions of number fields, building on the proof for quadratic extensions.

I. THE ELEMENTS OF POPESCU'S CONJECTURE

This article, representing work which was in progress at the time of the conference, presents evidence for Popescu's integral version of a Stark-type conjecture for abelian L-functions with higher order zeroes at the origin, in the case of an extension of number fields composed of relative quadratic extensions. We do not directly refer to other articles in this volume, but take the opportunity here to point to the article by Popescu for an illuminating introduction to this and related conjectures.

S-Class Groups and S-Units.

All fields are assumed to be contained in the field $\mathbb{C}$ of complex numbers.

Let:

- F be a fixed algebraic number field (finite extension of the rational numbers $\mathbb{Q}$).
- L/F be an abelian extension of number fields in which a fixed distinguished finite nonempty set $V = \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_r\}$ of (finite or infinite) primes of F splits completely. Note that $r \geq 1$ is defined to be the cardinality of V.
- $S = \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots \mathfrak{v}_r, \ldots, \mathfrak{v}_{r'}\}$ be a fixed set of primes of F which contains V, all of the infinite primes of F, and all of the primes which ramify in L. We assume that the cardinality of S is $|S| = r' > r + 1$, and write $r' = r + d + 1$. (The conjecture is actually made for $|S| = r + 1$ as well, but this tends to be a technical special case, and we will avoid it for the sake of simplicity.)
- S_L be the set of primes of L lying above those in S.
- $\mathrm{Cl}_F = \mathrm{Cl}_{F,S}$ be the S-ideal class group of F, of order $h_F = h_{F,S}$. This group is defined to be the quotient of the ideal class group of F by the subgroup generated by the classes represented by the finite primes in S.

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- $\mathrm{Cl}_L = \mathrm{Cl}_{L,S}$ be the S_L-ideal class group of L, of order $h_L = h_{L,S}$
- $r_F = r_F(S)$ be the 2-rank of $\mathrm{Cl}_{F,S}$, so $2^{r_F(S)} = |\mathrm{Cl}_{F,S}/\mathrm{Cl}^2_{F,S}|$
- $|\ |_{\mathfrak{w}_i}$ be the normalized absolute value at a fixed prime $\mathfrak{w}_i$ of L above $\mathfrak{v}_i$, for each $i = 1, 2, \ldots, r'$.
- w_L be the order of the group μ_L of roots of unity in L
- $U_L = U_{L,S}$ be the group of S-units (or more precisely, S_L-units) of L, defined as all elements of L having absolute value equal to 1 at each absolute value of L except for those associated with primes in S_L, ie., those which are conjugates of $|\ |_{\mathfrak{w}_i}$ for some i in $\{1, \ldots, r'\}$.

If $\mathfrak{w}$ is a prime of L and $\alpha \in L$, then the normalized absolute value $|\alpha|_{\mathfrak{w}}$ is defined as follows. When $\mathfrak{w}$ is a finite prime with residue field of cardinality $N\mathfrak{w}$ and α has valuation $\mathrm{ord}_{\mathfrak{w}}(\alpha)$, then $|\alpha|_{\mathfrak{w}} = N\mathfrak{w}^{-\mathrm{ord}_{\mathfrak{w}}(\alpha)}$. When $\mathfrak{w}$ is an infinite prime, $|\alpha|_{\mathfrak{w}}$ is the usual absolute value of the image of α in $\mathbb{R}$ determined by $\mathfrak{w}$ if $\mathfrak{w}$ is real, and it is the square of the usual absolute value of the image of α in $\mathbb{C}$ if $\mathfrak{w}$ is complex.

L-functions.

See [**11**] for further background and references. Let:

- G be the abelian Galois group of the extension L/F,
- $\hat{G}$ be the character group of G.
- $\mathfrak{p}$ run through the finite primes of F not in S
- $\mathfrak{a}$ run through integral ideals of F, prime to the elements of S
- $\mathbf{N}\mathfrak{a}$ denote the absolute norm of the ideal $\mathfrak{a}$
- $\sigma_{\mathfrak{a}} \in G$ be the well-defined automorphism attached to $\mathfrak{a}$ via the Artin map

For each $\chi \in \hat{G}$, we have the Artin L-function with Euler factors at the primes in S removed:

- $L(s,\chi) = L_S(s,\chi) = \sum\limits_{\substack{\mathfrak{a}\ \text{integral} \\ (\mathfrak{a},S)=1}} \dfrac{\chi(\sigma_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^s} = \prod\limits_{\text{prime}\ \mathfrak{p}\notin S} \big(1 - \dfrac{\chi(\sigma_{\mathfrak{p}})}{\mathbf{N}\mathfrak{p}^s}\big)^{-1}$ It is known that $L_S(s,\chi)$ has an analytic continuation and a functional equation relating it to $L_S(1-s,\chi^{-1})$. The order of its zero at $s=0$ is
- $r(\chi) = r_S(\chi) = \begin{cases} |S|-1 \text{ or} \\ |\{\mathfrak{q} \in S : \mathfrak{q} \text{ splits completely in field fixed} \\ \text{by the kernel of } \chi\}| \end{cases}$

  depending on whether or not χ is the trivial character χ_0.

We see that $r(\chi) \geq r$ for each $\chi \in \hat{G}$. Let:

- $L^{(r)}(0,\chi) = L_S^{(r)}(0,\chi) = \lim\limits_{s\to 0} L_S(s,\chi)/s^r$

Regulators.

For more details, see [**4**], [**5**], [**6**] and [**8**].

Let:

- $W = (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_r)$ be the r-tuple of primes of L chosen to lie above those in V.
- $(u_1, u_2, \ldots, u_r)$ be an r-tuple of elements of $U_{L,S}$.
- $\mathcal{R}_{L,W}(u_1,\ldots,u_r) = \mathcal{R}_L(u_1,\ldots,u_r) = \det\big(-\sum\limits_{\gamma\in G} \log|u_i^{\gamma}|_{\mathfrak{w}_j}\gamma^{-1}\big)$,

  the generalized regulator with values in the group ring $\mathbb{R}[G]$

Then $\mathcal{R}_{L,W}$ factors through the rth exterior power $\wedge^r U_L$ of U_L as a $\mathbb{Z}[G]$-module. We also use $\mathcal{R}_L = \mathcal{R}_{L,W}$ to denote the induced map $\wedge^r U_L \to \mathbb{R}[G]$, and note that it is a $\mathbb{Z}[G]$-module homomorphism. Extending $\mathbb{C}$-linearly gives a $\mathbb{C}[G]$-module homomorphism $\mathbb{C} \otimes_{\mathbb{Z}} \wedge^r U_L = \mathbb{C} \wedge^r U_L \to \mathbb{C}[G]$, still denoted by $\mathcal{R}_{L,W}$ or just $\mathcal{R}_L$. For $\chi \in \hat{G}$, we also extend $\mathbb{C}$-linearly to obtain a $\mathbb{C}$-algebra homomorphism $\chi : \mathbb{C}[G] \to \mathbb{C}$.

Now let:

- $e_\chi = \dfrac{1}{|G|} \displaystyle\sum_{\gamma \in G} \chi(\gamma)\gamma^{-1}$ be the primitive idempotent of $\mathbb{C}[G]$ associated with a character $\chi \in \hat{G}$.
- χ^{-1} be the complex conjugate character of χ.
- $\varepsilon_L = \varepsilon_{L/F} = \varepsilon_{L/F,S}$ be the unique element of $\mathbb{C} \wedge^r U_{L_S}$ such that
  1) $e_\chi \varepsilon_L = 0$ for each $\chi \in \hat{G}$ with $r_S(\chi) > r$, and
  2) $\chi^{-1}(\mathcal{R}_L(\varepsilon_L)) = L^{(r)}(0, \chi)$ for each $\chi \in \hat{G}$.

The existence of $\varepsilon_{L/F,S}$ is guaranteed by Remark 2 in section 2 of [**5**]. We will refer to property 1) as the "eigenspace property" and property 2) as the "evaluation property" of $\varepsilon_{L/F,S}$. Note that property 1) implies that $e_{\chi^{-1}}\chi^{-1}(\mathcal{R}_L(\varepsilon_L)) = e_{\chi^{-1}}\mathcal{R}_L(\varepsilon_L) = \mathcal{R}_L(e_{\chi^{-1}}\varepsilon_L) = 0$, so $\chi^{-1}(\mathcal{R}_L(\varepsilon_L)) = 0 = L^{(r)}(0, \chi)$, which is property 2) for these χ.

The Conjectures.

We note that F, V, W and S will remain fixed throughout so that we can safely omit references to them in the notation. Thus $U_{L,S} = U_L$, $L_S(s, \chi) = L(s, \chi)$, $\mathcal{R}_{L,W} = \mathcal{R}_L$, etc.

We have defined $\varepsilon_{L/F,S} \in \mathbb{C} \wedge^r U_{L,S}$ above. Stark's conjectures are concerned with the rationality and integrality properties of such elements. The rationality conjecture of [**9**] and [**11, Conjecture I.5.1**] applies to Artin L-functions in general. In our setting devoted to abelian L-functions, basic results of Rubin [**6, Propositions 2.3 and 2.4**] and Popescu [**5, Theorem 5.5.1**] show that the assertion that this conjecture is true for all $\chi \in \hat{G}$ with $r_S(\chi) = r$ is equivalent to the following conjecture.

Conjecture $St(L/F, S, r)$. $\varepsilon_{L/F,S} \in \mathbb{Q} \wedge^r U_{L,S}$

Stark [**10**] formulated an integrality conjecture for abelian L-functions when $r = 1$. To state Popescu's generalization of it we let

- $U_{L,S}^{\mathrm{ab}} = \{u \in U_{L,S} : L(u^{1/w_L})/F \text{ is abelian}\}$
- $\widetilde{U_{L,S}^{\mathrm{ab}}}$ denote its image in $\mathbb{Q}U_{L,S} = \mathbb{Q} \otimes_{\mathbb{Z}} U_{L,S}$, written additively.

Also let

- $$\wedge_0^r U_{L,S} = \{u = \sum_{t=1}^{m} c_t \left(u_1^{(t)} \wedge \cdots \wedge u_r^{(t)}\right) \in \mathbb{Q} \wedge^r U_{L,S} \ : \ \sum_{t=1}^{m} c_t \sum_{k=1}^{r} (-1)^k \left(\det_{i \neq r, j \neq k} \varphi_i(u_j^{(t)})\right) \cdot u_k^{(t)} \in \frac{1}{w_L}\widetilde{U_{L,S}^{\mathrm{ab}}} \quad \forall \varphi_1, \dots, \varphi_{r-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S}, \mathbb{Z}[G])\}$$

Now we can state Popescu's conjecture.

Conjecture $C(L/F, S, r)$. $\varepsilon_{L/F,S} \in \wedge_0^r U_{L,S}$

We will need to refer to a weaker version of this conjecture, in which $\widetilde{U_{L,S}^{\mathrm{ab}}}$ is replaced by $\widetilde{U_{L,S}}$, the image of $U_{L,S}$ in $\mathbb{Q}U_{L,S}$. Define

- $$\wedge_1^r U_{L,S} = \{u = \sum_{t=1}^{m} c_t \left(u_1^{(t)} \wedge \cdots \wedge u_r^{(t)}\right) \in \mathbb{Q} \wedge^r U_{L,S} \ :$$
$$\sum_{t=1}^{m} c_t \sum_{k=1}^{r} (-1)^k \left(\det_{i\neq r, j\neq k} \varphi_i(u_j^{(t)})\right) \cdot u_k^{(t)} \in \frac{1}{w_L}\widetilde{U_{L,S}}$$
$$\forall \varphi_1, \ldots, \varphi_{r-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{L,S}, \mathbb{Z}[G])\}$$

Since $U_{L,S}^{\mathrm{ab}} \subset U_{L,S}$, conjecture $C(L/F,S,r)$ implies the following "integrality conjecture without an abelian condition".

Conjecture $C'(L/F,S,r)$. $\varepsilon_{L/F,S} \in \wedge_1^r U_{L,S}$

II. STATEMENTS OF RESULTS

Theorem 2.1. *Conjecture* $C'(L/F,S,r)$ *holds when* $|S| > r+1$ *and* $G = \mathrm{Gal}(L/F)$ *has exponent 2.*

Theorem 2.2. *Conjecture* $C(L/F,S,r)$ *holds when* $|S| > r+1$ *and* G *has exponent 2 and order* 2^m*, provided that* $|S| + r_F(S) \geq r+m+1$ *and, when* $|S| + r_F(S) \leq r+m+2$*, we have* $\sqrt{-1} \notin L$.

Theorem 2.3. *Conjecture* $C(L/F,S,r)$ *holds when* $|S| > r+1$ *and* G *has exponent 2, provided that* L/F *is tame, i.e., unramified at finite primes of residue characteristic 2 (also known as dyadic primes), some infinite prime ramifies in* L/F*, and, when only one or two infinite primes ramify in* L/F*,* $\sqrt{-1} \notin L$.

Theorem 2.4. *Suppose that* $|S| > r+1$*,* G *has exponent 2, and* $S' = S \cup P$ *where* P *is a nonempty finite set of primes of* F *disjoint from* S*. If* $|P| = 1$*, assume that* $\sqrt{-1} \notin L$*. Then Conjecture* $C(L/F,S',r)$ *holds.*

Remark. *Under our assumption that Gal*(L/F) *has exponent 2, one can see fairly readily that "Conjecture C(L/F,S,r) holds after tensoring with* $\mathbb{Z}[\frac{1}{2}]$*", i.e. that* $2^t\varepsilon_{L/F,S} \in \wedge_0^r U_{L,S}$*, for some positive integer* t*. This will be apparent in the very beginning of the proof of Theorem 2.1, for example. Hence the focus is on carefully analyzing the factors of 2 which arise.*

Sections III, IV, and V will provide the ingredients which will allow us to prove these as well as a more general but more technical result (Proposition 6.2) in Section VI.

III. RELATIVE QUADRATIC EXTENSIONS

In this section, K/F will be a relative quadratic extension with $K \subset L$, and S a set of primes of F as above. Indeed, we can assume $L = K$ for the purposes of this section, but we will want the freedom to consider larger L later.

We know that Popescu's conjecture holds for relative quadratic extensions, because Popescu [**5,Theorem 5.5.1**] has shown that it follows from Rubin's conjecture of [**6**], while Rubin [**6, Section 3**] has proved his conjecture for relative quadratic extensions. However, we will need an explicit description of $\varepsilon_{K/F}$, so we derive this briefly. Our development follows that in [**6**], and indeed mostly results from choosing T to be the empty set there.

Let:

- $\overline{G} = \mathrm{Gal}(K/F) = \langle \tau \rangle$ of order 2.
- $\psi_0 \in \hat{\overline{G}}$ be the trivial character.
- $\psi \in \hat{\overline{G}}$ be the non-trivial character.
- $U_K = U_{K,S_K}$, the S_K-units of K.
- R_K and R_F be the S_K-regulator of U_K and the S-regulator of U_F, respectively.

Assume for the moment that S has exactly r primes which split completely in K, namely those in V. Then U_F has rank $r' - 1$ and U_K has rank $r' + r - 1$. Also $r_S(\psi) = r$, while $r_S(\psi_0) = r' - 1 > r$. Now let

- $u_1, \dots u_r$ in U_K generate $U_K/U_F\mu_K$ modulo its torsion subgroup $\mathrm{Tor} = \mathrm{Tor}_K$.
- $Q = Q_K = |\mathrm{Tor}_K|$
- $\eta = \eta_{K/F} = u_1^{1-\tau} \wedge \cdots \wedge u_r^{1-\tau}$ in $\wedge^r U_K$

Then the arguments of [**6, Theorem 3.5**] or [**7, Theorem 6.7**] show that (after replacing u_1 by u_1^{-1} if necessary)

$$\frac{R_K}{R_F} = \frac{2^d}{Q}\det(-\{\log|u_i|_{\mathfrak{w}_j} - \log|u_i^{\tau}|_{\mathfrak{w}_j}\}) = \frac{2^d}{Q}\det(-\{\log|u_i^{1-\tau}|_{\mathfrak{w}_j}\})$$
$$= \frac{2^d}{2^r Q}\det(-\{\log|u_i^{1-\tau}|_{\mathfrak{w}_j} - \log|(u_i^{1-\tau})^{\tau}|_{\mathfrak{w}_j}\}) = \frac{2^d}{2^r Q}\psi^{-1}(\mathcal{R}_K(\eta)),$$

the factor of 2^d arising from the fact that $|u|_{\mathfrak{w}}^2 = |u|_{\mathfrak{v}}$ when $u \in F$ and $\mathfrak{w}$ is the unique prime of K lying over a prime $\mathfrak{v}$ of F. Thus, from the generalized analytic class number formula (see [**6**] and [**3**]),

$$L^{(r)}(0,\psi) = \frac{w_F}{w_K}\frac{h_K}{h_F}\frac{R_K}{R_F} = \frac{2^d}{2^r Q}\frac{w_F}{w_K}\frac{h_K}{h_F}\psi^{-1}(\mathcal{R}_K(\eta)).$$

Also, for ψ_0, we have $r_S(\psi_0) > r$ and $e_{\psi_0}\eta = 0$ since $e_{\psi_0}u_1^{1-\tau} = (e_{\psi_0})(1-\tau)u_1 = 0u_1 = 0$. These properties of η show that we have established the following fact.

Proposition 3.1. *Under the assumption that* $|S| > r+1$, *we have* $\varepsilon_{K/F} = \frac{2^d}{2^r Q_K}\frac{w_F}{w_K}\frac{h_K}{h_F}\eta \in \mathbb{Q}\wedge^r U_{K,S}$ *for the relative quadratic extension* K/F *when exactly* r *primes of* S *split in* K, *and* $\varepsilon_{K/F} = 0$ *when more than* r *primes of* S *split in* K.

We now derive an equivalent expression for $\varepsilon_{K/F}$ which will be useful in the sequel. Still assuming that exactly r primes of S split in K, let

- $U_K^- = \{u \in U_K : u^{1+\tau} = 1\}$
- $v_1, \dots, v_r$ be a basis for U_K^- modulo torsion.

Note that the rank of U_K^- is r since it is the kernel of a homomorphism from U_K, of rank $r' + r - 1$, onto a subgroup of U_F which contains $U_F^2 = U_F^{1+\tau}$, hence is of rank $r' - 1$.

Since $u_1, \dots, u_r$ represent a basis for $U_K/\mu_K U_F$ modulo torsion, it is easy to see that $u_1^{1-\tau}, \dots, u_r^{1-\tau}$ represent a basis of $U_K^{1-\tau}/\mu_K^{1-\tau}$ modulo torsion, which is just $U_K^{1-\tau}$ modulo torsion. The map $1-\tau$ also gives an isomorphism between the torsion subgroup Tor_K of $U_K/\mu_K U_F$ and the torsion subgroup $\mu_K^- \cap U_K^{1-\tau}/\mu_K^{1-\tau}$ of $U_K^{1-\tau}/\mu_K^{1-\tau}$. So the order of the latter torsion subgroup is also Q. Now we can rewrite

$$\eta = u_1^{1-\tau} \wedge \cdots \wedge u_r^{1-\tau} = \pm(U_K^- : U_K^{1-\tau}\mu_K^-)v_1 \wedge \cdots \wedge v_r \in \mathbb{Q}\wedge^r U_{K,S},$$

and we compute

$$(U_K^- : U_K^{1-\tau}\mu_K^-) = (U_K^- : U_K^{1-\tau})/(U_K^{1-\tau}\mu_K^- : U_K^{1-\tau}) =$$
$$(U_K^- : U_K^{1-\tau})/(\mu_K^- : \mu_K^- \cap U_K^{1-\tau}) = (U_K^- : U_K^{1-\tau})(\mu_K^- \cap U_K^{1-\tau} : \mu_K^{1-\tau})/(\mu_K^- : \mu_K^{1-\tau}) =$$
$$Q|H^1(\overline{G}, U_K)|/|H^1(\overline{G}, \mu_K)|$$

So

$$\eta = Q\frac{|H^1(\overline{G}, U_K)|}{|H^1(\overline{G}, \mu_K)|}v_1 \wedge \cdots \wedge v_r$$

and

$$\varepsilon_{K/F} = \frac{2^d}{2^r Q}\frac{w_F}{w_K}\frac{h_K}{h_F}\eta = \frac{1}{w_K}\frac{2^d}{2^r}\frac{w_F}{|H^1(\overline{G}, \mu_K)|}\frac{|H^1(\overline{G}, U_K)|h_K}{h_F}v_1 \wedge \cdots \wedge v_r$$

Now since μ_K is finite, we have

$$|H^1(\overline{G}, \mu_K)| = |\hat{H}^0(\overline{G}, \mu_K)| = |\mu_F/N_{K/F}(\mu_K)| = \frac{w_F}{|N_{K/F}(\mu_K)|}$$

Also, one knows from the proof of [**11, Theorem IV.5.4**] that

$$\frac{|H^1(\overline{G}, U_K)|h_K}{h_F} = |\mathrm{Cl}_K/\iota\mathrm{Cl}_F|,$$

the order of the cokernel of the natural map ι from Cl_F to Cl_K induced by extension of ideals. We have arrived at the formula we sought.

PROPOSITION 3.2.
Let $M_K = |\mathrm{Cl}_K/\iota\mathrm{Cl}_F|$. When K/F is a relative quadratic extension, $|S| > r+1$ and the number of primes in S which split in K/F is exactly r, we have

$$\varepsilon_{K/F} = \frac{|N_{K/F}(\mu_K)|}{w_K}\frac{2^d}{2^r}M_K v_1 \wedge \cdots \wedge v_r$$

Next we analyze the factor $M_K = |\mathrm{Cl}_K/\iota\mathrm{Cl}_F| = |Cl_{K,S}/\iota\mathrm{Cl}_{F,S}|$.

PROPOSITION 3.3. *When K/F is a relative quadratic extension and the number of primes in S which split in K/F is exactly r, the norm map on ideals induces a surjective homomorphism $\mathrm{Cl}_K/\iota\mathrm{Cl}_F \to \mathrm{Cl}_F/\mathrm{Cl}_F^2$, and thus $2^{r_F} = |\mathrm{Cl}_F/\mathrm{Cl}_F^2|$ divides M_K.*

PROOF. By assumption, S contains a prime, namely $\mathfrak{v}_{r+1}$, which does not split in K/F. If H_F (resp. H_K) is the class field of F (resp. K) corresponding to $\mathrm{Cl}_F = \mathrm{Cl}_{F,S}$ (resp. $\mathrm{Cl}_K = \mathrm{Cl}_{K,S}$), then all primes of S split in H_F. Thus $H_F \cap K = F$, and the restriction map from $\mathrm{Gal}(H_K/K)$ to $\mathrm{Gal}(H_F/F)$ is surjective. By class field theory, this translates into the statement that the homomorphism from Cl_K to Cl_F induced by the norm is surjective. Our conclusion follows by composing this homomorphism with the projection onto $\mathrm{Cl}_F/\mathrm{Cl}_F^2$ and simply noting that $\iota\mathrm{Cl}_F$ is contained in the kernel of this composition. □

IV. PROPERTIES OF THE REGULATOR

In this section, we consider a tower of fields $F \subset K \subset L$ and suppose that we have fixed a choice of a prime $\mathfrak{w}_i$ of L above the prime $\mathfrak{v}_i$ of F for each $i = 1, \dots r$. The r-tuple $W = (\mathfrak{w}_1, \dots, \mathfrak{w}_r)$ was used to define the regulator map $\mathcal{R}_L = \mathcal{R}_{L,W}$ for L/F. Then we naturally define $\overline{\mathfrak{w}}_i$ to be the prime below $\mathfrak{w}_i$ in K, and use $\overline{W} = (\overline{\mathfrak{w}}_1, \dots, \overline{\mathfrak{w}}_r)$ to define the regulator map $\mathcal{R}_K = \mathcal{R}_{K,\overline{W}}$.

Now let $H = \mathrm{Gal}(L/K)$ so that $\overline{G} = \mathrm{Gal}(K/F) \cong G/H$, and let $\pi = \pi_{L/K}$ be the restriction map $G \to \overline{G}$. For $\gamma \in G$, we write $\overline{\gamma} = \pi(\gamma)$. Then π extends to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \to \mathbb{C}[\overline{G}]$.

The norm map $N : U_L \to U_K$ induces $N^{(r)} = \wedge^r N : \wedge^r U_L \to \wedge^r U_K$.

Lemma 4.1. *For $u \in \mathbb{C} \wedge^r U_L$, we have*

$$\pi_{L/K}(\mathcal{R}_L(u)) = \mathcal{R}_K(N^{(r)}(u)).$$

Proof. By $\mathbb{C}$-linearity, it suffices to prove equality when $u = u_1 \wedge \cdots \wedge u_r$, with each $u_i \in U_L$. Then

$$\pi(\mathcal{R}_L(u)) = \pi\big(\det\big(-\sum_{\gamma \in G} \log|u_i^{\gamma}|_{w_j} \gamma^{-1}\big)\big) = \det\big(-\sum_{\gamma \in G} \log|u_i^{\gamma}|_{w_j} \overline{\gamma}^{-1}\big)$$

$$= \det\big(-\sum_{\sigma \text{ rep } G/H} \sum_{\tau \in H} \log|u_i^{\sigma\tau}|_{w_j} \overline{\sigma}^{-1}\big) = \det\big(-\sum_{\sigma \text{ rep } G/H} \log|N(u_i)^{\sigma}|_{w_j} \overline{\sigma}^{-1}\big)$$

$$= \det\big(-\sum_{\overline{\sigma} \in \overline{G}} \log|N(u_i)^{\overline{\sigma}}|_{\overline{w_j}} \overline{\sigma}^{-1}\big) = \mathcal{R}_K(Nu_1, \dots, Nu_r) = \mathcal{R}_K(N^{(r)}(u)) \qquad \square$$

Corollary 4.2. *Suppose that $u \in \mathbb{C} \wedge^r U_L$ is the image of $u_K \in \wedge^r U_K$. Then*

$$\pi_{L/K}(\mathcal{R}_L(u)) = (L:K)^r \mathcal{R}_K(u_K).$$

Proof. By $\mathbb{C}$-linearity again, we may assume that $u_K = u_1 \wedge \cdots \wedge u_r$, with each $u_i \in U_K$. Then by the Lemma,

$$\pi_{L/K}(\mathcal{R}_L(u)) = \mathcal{R}_K(N^{(r)}(u)) = \mathcal{R}_K(N^{(r)}(u_1 \wedge \cdots \wedge u_r)) =$$
$$\mathcal{R}_K(N(u_1) \wedge \cdots \wedge N(U_r)) = \mathcal{R}_K(u_1^{(L:K)} \wedge \cdots \wedge u_r^{(L:K)}) =$$
$$\mathcal{R}_K((L:K)^r u_K) = (L:K)^r \mathcal{R}_K(u_K). \qquad \square$$

Corollary 4.3.

$$N^{(r)}(\varepsilon_{L/F}) = \varepsilon_{K/F}$$

Proof.

First, for $\chi \in \hat{\overline{G}}$, let $\tilde{\chi} = \chi \circ \pi$ denote its inflation to $\hat{G}$. Then using the lemma and the inflation property of Artin L-functions, we have:

$$\chi(\mathcal{R}_K(N^{(r)}(\varepsilon_{L/F}))) = \chi(\pi(\mathcal{R}_L(\varepsilon_{L/F}))) = \tilde{\chi}(\mathcal{R}_L(\varepsilon_{L/F})) =$$
$$L^{(r)}(0, \tilde{\chi}^{-1}) = L^{(r)}(0, \chi^{-1}).$$

Since this holds for all χ, we see that $N^{(r)}(\varepsilon_{L/F})$ satisfies the evaluation property of $\varepsilon_{K/F}$.

To check the eigenspace property of $\varepsilon_{K/F}$, we consider χ such that $r_S(\chi) > r$ and show that $e_\chi N^{(r)}(\varepsilon_{L/F}) = 0$. Now $r_S(\chi) = r_S(\tilde{\chi})$, so $e_{\tilde{\chi}}\varepsilon_{L/F} = 0$. Hence $0 = N^{(r)} e_{\tilde{\chi}}\varepsilon_{L/F} = e_\chi N^{(r)}\varepsilon_{L/F}$ in $\mathbb{C} \wedge^r U_K$. □

Now we fix the setting which will be of most interest to us for the rest of this paper. From now on:

- $G = \mathrm{Gal}(L/F)$ will be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$, for a fixed positive integer m
- K_i for $i = 1, \ldots, 2^m - 1$ will be the quadratic extensions of F in L
- $\pi_i = \pi_{L/K_i}$
- $\varepsilon_i = \varepsilon_{K_i/F}$, and use the same symbol for its image in $\mathbb{C} \wedge^r U_L$.
- $\varepsilon = \dfrac{1}{(2^{m-1})^r}[-(2^m - 2)2^{-r}\varepsilon_{F/F} + (\varepsilon_1 + \cdots + \varepsilon_{2^m-1})]$

Here L/F is what we call a multi-quadratic extension: the field L is the composite of (m of) the quadratic extensions K_i of F.

Remark. *Under our standing assumption that $|S| > r + 1$, we actually have $\varepsilon_{F/F} = 0$. The results of this section are written so as to be valid even when $|S| = r + 1$.*

Corollary 4.4.

$$\pi_i(\mathcal{R}_L(\varepsilon)) = \mathcal{R}_{K_i}(\varepsilon_i).$$

Proof. Let $N_i = N_{L/K_i}$. Using Lemma 4.1, we have

$$\begin{aligned}(2^{m-1})^r\pi_i(\mathcal{R}_L(\varepsilon)) = (2^{m-1})^r\mathcal{R}_{K_i}(N_i^{(r)}(\varepsilon)) = \\ -(2^m - 2)2^{-r}\mathcal{R}_{K_i}(N_i^{(r)}(\varepsilon_{F/F})) + \sum_j \mathcal{R}_{K_i}(N_i^r(\varepsilon_j)).\end{aligned}$$

Note that for $j = i$,

$$\mathcal{R}_{K_i}(N_i^{(r)}(\varepsilon_i)) = \mathcal{R}_{K_i}((L:K_i)^r\varepsilon_i) = (L:K_i)^r\mathcal{R}_{K_i}(\varepsilon_i) = (2^{m-1})^r\mathcal{R}_{K_i}(\varepsilon_i)$$

Now consider a term with $j \neq i$. In this case, we have

$$\begin{aligned}N_i^{(r)}(\varepsilon_j) = N^{(r)}_{L/K_jK_i}N^{(r)}_{K_jK_i/K_i}(\varepsilon_j) = N^{(r)}_{L/K_jK_i}N^{(r)}_{K_j/F}(\varepsilon_j) = \\ N^{(r)}_{L/K_jK_i}(\varepsilon_{F/F}) = (2^{m-2})^r(\varepsilon_{F/F}),\end{aligned}$$

so altogether these terms contribute $(2^m - 2)(2^{m-2})^r\mathcal{R}_{K_i}(\varepsilon_{F/F})$. Finally, the remaining term contributes

$$\begin{aligned}-(2^m - 2)2^{-r}\mathcal{R}_{K_i}(N_i^{(r)}(\varepsilon_{F/F})) = -(2^m - 2)2^{-r}\mathcal{R}_{K_i}((2^{m-1})^r\varepsilon_{F/F}) \\ = -(2^m - 2)2^{-r}(2^{m-1})^r\mathcal{R}_{K_i}(\varepsilon_{F/F}) = (2^m - 2)(2^{m-2})^r\mathcal{R}_{K_i}(\varepsilon_{F/F}),\end{aligned}$$

leaving the desired result. □

Proposition 4.5. $\varepsilon = \varepsilon_{L/F}$.

Proof. Let $\tilde{\chi} \in \hat{G}$. The image of $\tilde{\chi}$ is cyclic, so the kernel of $\tilde{\chi}$ contains one of the subgroups $H_i = \mathrm{Gal}(L/K_i)$ fixing a quadratic extension K_i of F in L. Thus $\tilde{\chi}$ is the inflation of a character χ on G/H_i. That is, $\tilde{\chi} = \chi \circ \pi_i$, where π_i is the natural projection of G on G/H_i. Then using Corollary 4.4 and the inflation property of Artin L-functions, we have

$$\tilde{\chi}(\mathcal{R}_L(\varepsilon)) = \chi(\pi_i(\mathcal{R}_L(\varepsilon))) = \chi(\mathcal{R}_{K_i}(\varepsilon_i)) = L^{(r)}(0, \chi^{-1}) = L^{(r)}(0, \tilde{\chi}^{-1}).$$

This holds for all $\tilde{\chi} \in \hat{G}$, so ε satisfies the evaluation property of $\varepsilon_{L/F}$

Now we check the eigenspace property of $\varepsilon_{L/F}$. Suppose $\tilde{\chi} \in \hat{G}$ has $r(\tilde{\chi}) > r$. As above, we know that $\tilde{\chi} = \chi \circ \pi_i$ for some fixed i. Then $r(\chi) = r(\tilde{\chi}) > r$, so $e_\chi \varepsilon_i = 0$ in $\mathbb{C} \wedge^r U_{K_i}$ and hence its image in $\mathbb{C} \wedge^r U_L$ is $e_{\tilde{\chi}} \varepsilon_i = e_\chi \varepsilon_i = 0$. Now consider $e_{\tilde{\chi}} \varepsilon_j$ for $j \neq i$. Then we may choose τ not in H_j but in the kernel of $\tilde{\chi}$. Hence $1 + \tau$ acts as the norm $N_{K_j/F}$ on K_j. Then

$$2^r e_{\tilde{\chi}} \varepsilon_j = (1 + \tilde{\chi}(\tau))^r e_{\tilde{\chi}} \varepsilon_j = (1 + \tau)^r e_{\tilde{\chi}} \varepsilon_j = e_{\tilde{\chi}} (1 + \tau)^{(r)} \varepsilon_j = e_{\tilde{\chi}} \varepsilon_{F/F},$$

so $e_{\tilde{\chi}}(\varepsilon_j - 2^{-r} \varepsilon_{F/F}) = 0$. Summing this over all $j \neq i$ and combining with the term $2^r e_{\tilde{\chi}} \varepsilon_i = 0$ gives $2^r \varepsilon_{\tilde{\chi}} \varepsilon = 0$. □

Proposition 3.1 and Proposition 4.5 immediately imply the truth of the rational Stark Conjecture $St(L/F, S, r)$ of [**11, I.5.1**] for the multiquadratic extension L/F. This result is already known to be true, for example, because all characters of G are rational, and so the main theorem of [**9**] applies. Thus our goal is to prove the stronger, integral Stark-type Conjecture $C(L/F, S, r)$ of Popescu. We will do so under some additional hypotheses.

V. Kummer Theory

Our standing assumption is that K/F is a relative quadratic extension unramified outside of S in which all primes in $V \subset S$ split completely. The results of this section deal specifically with K for which exactly r primes in S split completely, namely those in V. Since S is fixed, U_K denotes the S-units of K, and U_K^- denotes the subgroup of elements u whose relative norm $u^{1+\tau}$ to F equals 1.

Lemma 5.1. *When exactly r primes of S split in K, we have*

$$|U_K^-/(U_K^-)^2| = 2^{r+1}.$$

Proof. We have observed in section 3 that the free rank of U_K^- is r. Clearly $-1 \in U_K^-$, so the cyclic torsion subgroup of U_K^- contains 2-torsion. Thus

$$|U_K^-/(U_K^-)^2| = 2^{r+1}.$$

□

Now let:

- $\mathfrak{m}_S = \prod_{i=1}^{r'} \mathfrak{v}_i$, a formal modulus for F defined as the product of all primes in S.
- $\mathcal{L} = \mathcal{L}_S$ be the composite of all quadratic extensions of F in $\mathbb{C}$ with relative discriminant dividing $4\mathfrak{m}_S$. This may also be described as the maximal multiquadratic extension of F with conductor dividing $4\mathfrak{m}_S$.

Now return to the setting of a tower of fields $F \subset K \subset L$, with L/F multiquadratic. Suppose that K is one of the relative quadratic extensions of F in L with the property that exactly r primes of S split in K.

Let:

- $2^a = |U_K^-/(U_K^- \cap U_L^2)|$
- $2^b = |(U_K^- \cap U_L^2)/(U_K^-)^2(\mu_K^- \cap \mu_L^2)|$

Then by Lemma 5.1,

$$2^{r+1} = |U_K^-/(U_K^-)^2| = 2^{a+b}|(U_K^-)^2(\mu_K^- \cap \mu_L^2)/(U_K^-)^2| = 2^{a+b}|(\mu_K^- \cap \mu_L^2)/(\mu_K^-)^2|$$

We record the following corollary for future reference.

Corollary 5.2.

$$2^{a+b} = 2^r|\mu_K^-/(\mu_K^- \cap \mu_L^2)|$$

Two other results we will need are straightforward modifications of those found in [**2**]; summarized in the following Proposition.

Proposition 5.3.

1. *The field* $\mathcal{L} = \mathcal{L}_S$ *contains* $L(\sqrt{U_K^-})$.
2. $[\mathcal{L}_S : F] = 2^{r_F(S)+|S|}$

Lemma 5.4. *Suppose that exactly* r *primes of* S *split in* K*. Then* $r_F(S)+|S| \geq a+m$*.*

Proof. By the first part of Proposition 5.3, $\mathcal{L}_S$ contains $L(\sqrt{U_K^-})$. We simply compute the degree of this relative extension and note that it equals $2^{r_F(S)+|S|-a-m}$. This will yield the desired inequality.

By the second part of Proposition 5.3, the degree of $\mathcal{L}_S$ over F is $2^{r_F(S)+|S|}$. The degree of L over F is 2^m. We now show that the degree of $L(\sqrt{U_K^-})/L$ is 2^a, from which it will follow that $2^{a+m} = [L(\sqrt{U_K^-}) : F]$ and the result will then be clear. But Kummer theory gives $[L(\sqrt{U_K^-}) : L] = |U_K^-(L^*)^2/(L^*)^2| = |U_K^-/(U_K^- \cap (L^*)^2)| = 2^a$, since $U_K^- \cap (L^*)^2 = U_K^- \cap U_L^2$. This last equality uses the property that $U_L \cap (L^*)^2 = U_L^2$, i.e., an S-unit which is a square, is a square of an S-unit. (Note that the situation would be different if we imposed congruence conditions on our units as well.) $\square$

VI. Proofs of Theorems

We now consider Popescu's conjecture for a fixed multiquadratic extension L/F and appropriate set S. By Proposition 4.5, $\varepsilon_{L/F}$ is a sum over K of terms of the form $\frac{1}{2^{mr-r}}\varepsilon_{K/F}$, where K runs through the relative quadratic extensions of F in L. However $\varepsilon_{K/F} = 0$ by Proposition 3.1 if more than r primes of S split in K. Thus we may sum over K with the property that exactly r primes of S split in K. We consider $\varepsilon_{L/F}$ one term at a time: fix one such K.

Now consider the image of $(U_K^-)^2$ inside of $U_K^- \cap U_L^2$ modulo torsion. It has index 2^b, so we may choose $z_1, \ldots, z_r$ in U_L such that $z_1^2, \ldots, z_r^2$ constitutes a basis for $U_K^- \cap U_L^2$ modulo torsion while $z_1^4, z_2^4, \ldots, z_b^4, z_{b+1}^2, \ldots z_r^2$ forms a basis for $(U_K^-)^2$

modulo torsion. If $v_1, \ldots, v_r$ is a basis for U_K^- as in section 3, then $v_1^2, \ldots, v_r^2$ forms a basis for $(U_K^-)^2$, so that

$$2^b z_1 \wedge \cdots \wedge z_r = \pm v_1 \wedge \cdots \wedge v_r$$

in $\mathbb{Q} \wedge^r U_L$, and we may assume the sign is positive by changing z_1 to z_1^{-1}.

By Proposition 3.2 and Corollary 5.2,

$$\begin{aligned} \varepsilon_K = \varepsilon_{K/F} &= \frac{|N_{K/F}(\mu_K)|}{w_K} 2^d M_K \frac{1}{2^r} v_1 \wedge \cdots \wedge v_r \\ = \frac{|\mu_K/\mu_K^-|}{w_K} 2^d M_K \frac{2^b}{2^r} z_1 \wedge \cdots \wedge z_r &= \frac{|\mu_K/\mu_K^-|}{w_K} 2^d M_K \frac{|\mu_K^-/(\mu_K^- \cap \mu_L^2)|}{2^a} z_1 \wedge \cdots \wedge z_r, \end{aligned}$$

so the corresponding term in $\varepsilon_{L/F}$ is

$$\begin{aligned} \frac{1}{2^{mr-r}} \varepsilon_{K/F} &= \frac{|\mu_K/(\mu_K^- \cap \mu_L^2)|}{w_K} 2^{d+r} M_K \frac{1}{2^{mr+a}} z_1 \wedge \cdots \wedge z_r \\ &= \frac{|\mu_L^2/(\mu_K^- \cap \mu_L^2)|}{w_L} \frac{2^{|S|} M_K}{2^{m+a}} \frac{1}{2^{m(r-1)}} z_1 \wedge \cdots \wedge z_r \end{aligned}$$

PROOF OF THEOREM 2.1. We will show that this term lies in $\wedge_1^r U_{L,S}$. As this holds for each K, we can then conclude that $\varepsilon_{L/F} \in \wedge_1^r U_{L,S}$, thus completing the proof of $C'(L/F, S, r)$ in this situation; this is exactly the statement of theorem 2.1. Under additional hypotheses, we will show that this term lies in $\wedge_0^r U_{L,S}$ for each K and thereby proving $C(L/F, S, r)$ and obtaining Proposition 6.2. Theorems 2.2 and 2.3 will then follow as straightforward consequences.

First, note that the factor $2^{|S|} M_K / 2^{m+a}$ is integral by Proposition 3.3 and Lemma 5.4. Hence to obtain $C'(L/F, S, r)$, it suffices to show that

$$(1/w_L) \frac{1}{2^{m(r-1)}} z_1 \wedge \cdots \wedge z_r \in \wedge_1^r U_{L,S}.$$

So let $\varphi_1, \ldots \varphi_{r-1} \in \mathrm{Hom}_{Z[G]}(U_{L,S}, Z[G])$. Our task is to show that

$$\frac{1}{w_L} \frac{1}{2^{m(r-1)}} \sum_{k=1}^{r} (-1)^k \left(\det_{i \neq r, j \neq k} \varphi_i(z_j) \right) \cdot z_k \in \frac{1}{w_L} \widetilde{U_{L,S}}.$$

In fact we claim that for each k, we have

$$\frac{1}{2^{m(r-1)}} \left(\det_{i \neq r, j \neq k} \varphi_i(z_j) \right) \cdot z_k \in \widetilde{U_{L,S}}.$$

With the use of the following proposition, we will show that this element is actually an integer multiple of z_k in $\widetilde{U_{L,S}}$. □

LEMMA 6.1. *Let* $\{\sigma_i : 1 \le i \le m-1\}$ *generate* $\mathrm{Gal}(L/K)$, *and let* $\tau \in \mathrm{Gal}(L/F)$ *restrict to the generator of* $\mathrm{Gal}(K/F)$. *Put* $\alpha = (1-\tau) \prod_{1 \le i \le m-1} (1 + \sigma_i) \in \mathbb{Z}[G]$. *Then for each* i *and* j, *we have* $\varphi_i(z_j) = n_{i,j} \alpha$ *for some integer* $n_{i,j}$.

Proof. We suppress the subscripts on φ and z. Put $\varphi(z) = \sum_{\sigma\in G} n_\sigma \sigma$. Note that $z^2 \in U_K^-$. Thus $1 = (z^2)^{1+\tau} = (z^{1+\tau})^2$, so $0 = \varphi((z^{1+\tau})^2) = 2(1+\tau)\varphi(z)$, and this implies that $0 = (1+\tau)\varphi(z)$. A clear consequence of this is that $n_{\sigma\tau} = -n_\sigma$ for each σ in G. Similarly, for each i, we have $1 = (z^2)^{1-\sigma_i} = (z^{1-\sigma_i})^2$, so $0 = \varphi((z^{1-\sigma_i})^2) = 2(1-\sigma_i)\varphi(z)$, and $0 = (1-\sigma_i)\varphi(z)$. Consequently $n_{\sigma\sigma_i} = n_\sigma$ for all σ in G, and each i. From these identities, one concludes that n_σ equals a constant integer n on $\mathrm{Gal}(L/K)$, and equals $-n$ on the other coset of $\mathrm{Gal}(L/K)$ in G. It is a straightforward check that this statement is equivalent to the conclusion of the Lemma. ☐

Conclusion of Proof of Theorem 2.1. Consider the $(r-1)$ by $(r-1)$ determinant $\det \varphi_i(z_j)$: we now know that we can factor α out of each row, yielding $(\det n_{i,j})\alpha^{r-1}$. We need to determine the action of this element on the image of z_k in $\widetilde{U_{L,S}}$.

Again, $z_k^2 \in U_K^-$ for some positive integer c_k, and hence $1 = (z_k^2)^{1+\tau} = (z_k^{1+\tau})^2$. Dividing z_k^4 by both sides of this equality gives

$$(z_k^2)^2 = (z_k^{1-\tau})^2,$$

which shows that $(1-\tau)\cdot z_k = 2\cdot z_k$ in $\widetilde{U_{L,S}}$. Similarly $(1+\sigma_i)\cdot z_k = 2\cdot z_k$ in $\widetilde{U_{L,S}}$ for $i = 1, 2, \ldots, m-1$, because σ_i fixes K. We conclude that $\alpha \cdot z_k = 2^m z_k$ and hence $\alpha^{r-1}\cdot z_k = (2^m)^{r-1} z_k$ in $\widetilde{U_{L,S}}$.

Finally

$$\frac{1}{2^{m(r-1)}}\big(\det_{i\neq r, j\neq k} \varphi_i(z_j)\big)\cdot z_k = \frac{1}{2^{m(r-1)}}\big(\det_{i\neq r, j\neq k} n_{i,j}\big)(2^m)^{r-1}\cdot z_k = \big(\det_{i\neq r, j\neq k} n_{i,j}\big)\cdot z_k,$$

which lies in $\widetilde{U_{L,S}}$, as desired. This completes the proof of Theorem 2.1. ☐

Proposition 6.2.

For each relative quadratic extension K of F in L having exactly r primes of S split in K, let

$$c_K = \frac{|\mu_K^-/(\mu_K^- \cap \mu_L^2)|}{2}\,\frac{2^{|S|-1}M_K}{2^{m+a}},$$

and suppose that either c_K is integral or c_K is half-integral and $a = r+1$. Then $C(L/F,S,r)$ holds.

Proof. From the proof of Theorem 2.1, we see that each of the two factors of c_K is either integral or half-integral. The current proof is similar to that of Theorem 2.1, except that we do not omit the integral factor $|\mu_L^2/(\mu_K^- \cap \mu_L^2)|\frac{2^{|S|}M_K}{2^{m+a}}$, and must show that we obtain an element of $\widetilde{U_{L,S}^{\mathrm{ab}}}$ rather than just $\widetilde{U_{L,S}}$. So it suffices to show that

$$\begin{aligned}|\mu_L^2/(\mu_K^- \cap \mu_L^2)|\frac{2^{|S|}M_K}{2^{m+a}} z_k &= \frac{w_L}{w_K}|\mu_K/\mu_K^-|\frac{|\mu_K^-/(\mu_K^- \cap \mu_L^2)|}{2}\,\frac{2^{|S|-1}M_K}{2^{m+a}} z_k^2 \\ &= \frac{w_L}{w_K}|\mu_K/\mu_K^-| c_K z_k^2\end{aligned}$$

lies in $\widetilde{U_{L,S}^{\mathrm{ab}}}$. Recall that, in $\widetilde{U_{L,S}}$, we have $z_k^2 = y_k$, where y_k is an element of U_K^-. Thus our element is the image in $\widetilde{U_{L,S}}$ of $y_k^{|\mu_K/\mu_K^-| c_K w_L/w_K}$.

When $a = r + 1$, we see in Corollary 5.2 that $b = 0$. This implies that z_k is already the image of an element $y_k \in U_K^-$, and we may instead simply take $y_k = z_k$ in $\widetilde{U_{L,S}}$. So in the case where c_K is half-integral, our element is the image of $y_k^{|\mu_K/\mu_K^-|(2c_K)w_L/w_K}$.

In either case, we claim that $y_k^{|\mu_K/\mu_K^-|w_L/w_K}$ lies in $U_{L,S}^{\mathrm{ab}}$, and the result clearly follows. So we show that the w_L-th root $y_k^{|\mu_K/\mu_K^-|/w_K}$ generates an abelian extension of L. Indeed, $y_k^{|\mu_K/\mu_K^-|/w_K}$ generates an abelian extension of K. We repeat the argument found in [**11, p. 105**]. Choose an integer n_τ such that $\tau - n_\tau$ annihilates μ_K, where τ is again the generator of $\mathrm{Gal}(K/F)$. By [**11, Proposition IV.1.2**], it suffices to check that $(y_k^{|\mu_K/\mu_K^-|})^{\tau-n_\tau} \in U_K^{w_K}$.

Now on μ_K^-, τ acts as -1. So $-1 - n_\tau$ annihilates μ_K^-, $|\mu_K/\mu_K^-|(-1 - n_\tau)$ annihilates μ_K, and consequently $|\mu_K/\mu_K^-|(-1 - n_\tau) = w_K t_K$ for some integer t_K. Finally, using the fact that $y_k \in U_K^-$, we have

$$(y_k^{|\mu_K/\mu_K^-|})^{\tau-n_\tau} = (y_k^{\tau-n_\tau})^{|\mu_K/\mu_K^-|} = y_k^{(-1-n_\tau)|\mu_K/\mu_K^-|} = y_k^{t_K w_K} \in U_K^{w_K}$$

This completes the proof. □

Proof of Theorem 2.2. We will apply Proposition 6.2. So fix a relative quadratic extension K of F in L with exactly r primes of S splitting in K, if such exists. We have observed that each of the two factors of c_K is integral or half-integral. We have also observed from Corollary 5.2 that $a \leq r + 1$

If $a = r + 1$ for this extension, we need only show that c_K is half-integral by showing that one of the factors is integral. The second factor $2^{|S|-1}M_K/2^{m+a}$ is an integer multiple of $2^{|S|-1+r_F(S)}/2^{m+a} = 2^{|S|-1+r_F(S)}/2^{m+r+1}$ by Proposition 3.3. Clearly this is integral and we are done unless $|S| - 1 + r_F(S) < m + r + 1$. In this eventuality, our assumption forces $|S| + r_F(S) = m + r + 1$, and we then have additional hypotheses. Under these hypotheses, we consider the first factor $|\mu_K^-/(\mu_K^- \cap \mu_L^2)|/2$ of c_K in the proposition and show that it is integral. The quotient group in question is clearly cyclic of exponent 2. Since clearly $-1 \in \mu_K^-$, the hypothesis $\sqrt{-1} \notin L$ implies that this group is non-trivial, and hence has order 2.

Now suppose that $a \leq r$. Our hypothesis $|S| - 1 + r_F(S) \geq m + r$ implies that the second factor $2^{|S|-1}M_K/2^{m+a}$ in Proposition 6.2 is an integer, by Proposition 3.3. Indeed it is twice an integer unless $a = r$. When it is twice an integer, this makes c_K integral, as the first factor is half-integral or integral. When it is not twice an integer, we must have $a = r$ and $|S| - 1 + r_F(S) = m + r$, in which case we have still made the additional assumptions which imply the integrality of the first factor. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Again we will apply Proposition 6.2, first fixing a relative quadratic extension K of F in L with exactly r primes of S (namely $\mathfrak{v}_1, \ldots, \mathfrak{v}_r$) splitting in K, if such exists.

Order the remaining $d + 1$ primes of S so that $\mathfrak{v}_{r+1}, \ldots, \mathfrak{v}_{r+d+1-t}$ are the finite primes of L/F, where $t \geq 1$ is the number of ramified infinite primes of L/F. (Recall that S contains all infinite primes of F.) The primes $\mathfrak{v}_{r+1}, \ldots, \mathfrak{v}_{r+d+1-t}$ have unique primes $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_{r+d}$ lying over them in K. For $1 \leq i \leq d + 1 - t$, let D_i be the decomposition group and I_i be the inertia group for $\mathfrak{v}_{r+i}$ in L/F. Now D_i/I_i is cyclic and hence has order at most 2, which is the exponent of $G = \mathrm{Gal}(L/F)$. Also I_i is cyclic by the assumption of tameness which means that the higher ramification groups are trivial. Thus I_i also has order 1 or 2, and D_i has order dividing 4. The decomposition group of $\mathfrak{v}_i$ in K/F is the restriction of D_i to K ([**1, p. 99**]), and this restriction is not trivial, since we know that $\mathfrak{v}_{r+i}$ does not split in K. Hence D_i is not contained in $\mathrm{Gal}(L/K)$, and $D_i' = D_i \cap \mathrm{Gal}(L/K) \neq D_i$. Thus the subgroup D_i' of D_i has order 1 or 2. But D_i' is the decomposition group of $\mathfrak{p}_{r+i}$ in L/K.

Let D be the subgroup generated by $D_1', \ldots D_{d+1-t}'$. Thus D has order dividing 2^{d+1-t}, Let L^D be the fixed field of D in L. Thus $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_{r+d+1-t}$ split completely in L^D/K and $[L : L^D] = |D|$ divides 2^{d+1-t}, so $[L^D : K]$ is an integer multiple of $2^{m-1-(d+1-t)} = 2^{m-d+(t-2)}$. The t infinite primes $\mathfrak{p}_{r+d+2-t}, \ldots \mathfrak{p}_{r+d+1}$ are necessarily complex, and so also split in L^D/K. Finally, the remaining primes of S_K lie over those in V which split completely in L/F, so these also split completely in L^D/K. We conclude that all primes of S_K split completely in L^D/K, and no primes ramify as S contains all ramified primes of L/F.

Thus L^D/K is an abelian unramified extension in which all primes above those in S split completely. By class field theory, the Artin map induces a homomorphism ϕ from $\mathrm{Cl}_{K,S}$ onto $\mathrm{Gal}(L^D/K)$. If $\mathfrak{v}$ is a prime ideal of F which is unramified in L/F, we know that the Artin map applied to its extension $\mathfrak{p}$ in K gives the square of the Frobenius of $\mathfrak{v}$ in L/F ([**1, p. 99**]). But every square in G is trivial, so $\iota(\mathrm{Cl}_{F,S})$ lies in the kernel of the homomorphism ϕ. Thus $M_K = |\mathrm{Cl}_{K,S}/\iota(\mathrm{Cl}_{F,S})|$ is an integer multiple of $|\mathrm{Gal}(L^D/K)|$, which is an integer multiple of $2^{m-d+(t-2)}$. In other words,

$$M_K/2^{m-d+(t-2)} = 2^{r+d}M_K/2^{m+r}2^{t-2} = 2^{|S|-1}M_K/2^{m+r}2^{t-2}$$

is an integer.

Thus when $t \geq 3$, the second factor $2^{|S|-1}M_K/2^{m+a}$ of c_K is even when $a \leq r$ and integral when $a = r+1$. Then Proposition 6.2 completes the proof in this case.

When $t = 2$, the factor $2^{|S|-1}M_K/2^{m+a}$ is integral when $a \leq r$ and integral or half-integral when $a = r+1$. The assumption that $\sqrt{-1} \notin L$ when $t = 2$ implies the integrality of the first factor of c_K as in the previous proof. The hypotheses of Proposition 6.2 are again fulfilled, and the proof completed in this case.

When $t = 1$, there is exactly one infinite prime ramified in K/F. Then, by an argument of Tate ([**11,p. 98**]), some finite prime, which we may assume is $\mathfrak{v}_{r+1}$, must ramify. For otherwise, -1 would be a local norm in K/F at every prime except $\mathfrak{v}_{r'}$, and hence also at $\mathfrak{v}_{r'}$, which would then have to be unramified. Then $K = F(\sqrt{\gamma})$, where $\gamma \in F$ must have an odd valuation at $\mathfrak{v}_{r+1}$ since K/F is tamely ramified at $\mathfrak{v}_{r+1}$. The product formula for local Artin maps in L/K ([**1, p. 189**]) applied to $\sqrt{\gamma}$ then shows that D_1' is in the subgroup generated by $D_2', \ldots, D_d'$, and hence D has order dividing 2^{d-1} in this case. The proof is then completed as in the case of $t = 2$. □

Proof of Theorem 2.4. The ramified and infinite primes of L/F are contained in S, so each prime in P is finite and unramified. Applying Proposition 6.2 for the set S', we need only consider K in which exactly r primes of S' split completely; hence the primes of P are inert in K/F. If $\mathfrak{v} \in P$, then the unique prime $\mathfrak{p}$ above it in K represents an element in $\iota\mathrm{Cl}_{F,S}$. Hence $\mathfrak{p}$ represents a trivial element in $\mathrm{Cl}_{K,S}/\iota\mathrm{Cl}_{F,S}$. The order of this group being $M_{K,S}$, we see that

$M_{K,S'} = M_{K,S}$. Note also that $U^-_{K,S'} = U^-_{K,S}$. This is because $u \in U^-_{K,S'}$ implies that $u^{1+\tau} = 1$, while τ fixes $\mathfrak{p}$, so $|u^2|_{\mathfrak{p}} = |u^{1+\tau}|_{\mathfrak{p}} = 1$. Thus the quantity a defined by $2^a = |U^-_K/(U^-_K \cap (L^*)^2|$ (see the proof of Lemma 5.4) is also unchanged when S is replaced by S'.

We have observed that $2^{|S|}M_K/2^{m+a}$ is integral. Now $|S'| > |S|$, so $2^{|S'|-1}M_K/2^{m+a}$ is integral, and indeed twice an integer if $|P| > 1$. The application of Proposition 6.2 to complete the proof now clearly proceeds as in the case of Theorem 2.2. $\square$

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