Algebraic Construction of Rotation Matrices

- Unit vectors
- Complete Orthogonal Systems and the Identity Matrix
- Constructing a rotation matrix for a change of basis.

Vector Notation

Let $\mathbf{a}$ and $\mathbf{b}$ denote two vectors in $\mathbb{R}^2$. Each can be represented as column vectors. Thus,

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The *transpose* of each is represented as a *row vector*:

$$\mathbf{a}^T = \begin{pmatrix} a_1 & a_2 \end{pmatrix}, \quad \mathbf{b}^T = \begin{pmatrix} b_1 & b_2 \end{pmatrix}.$$

The *inner product* of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

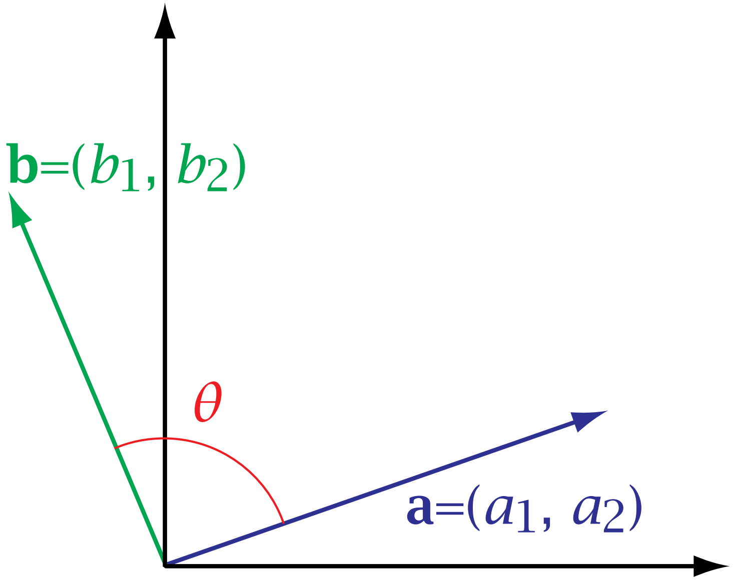
Observe that $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$.

The *length* of $\mathbf{a}$ is defined by $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$. (The length of a vector can also be denoted by symbol without a subscript. Thus, $a = \|\mathbf{a}\|$.

Observe that $\mathbf{a}^T\mathbf{a} = a_1^2 + a_2^2 = \|\mathbf{a}\|^2 = a^2$.

Inner Product: Geometric Version

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors that are separated by an angle θ, then their inner product can be alternatively defined as $\mathbf{a}^T\mathbf{b} = ab\cos\theta$



$$\mathbf{a}^T\mathbf{b} = a_1b_1 + a_2b_2$$

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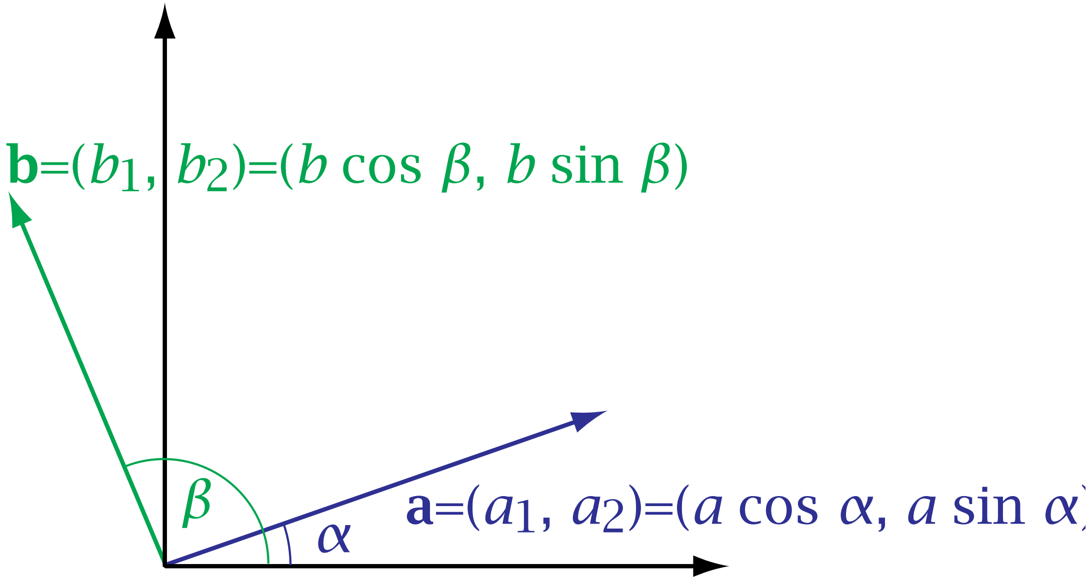

$$
\begin{aligned}
\mathbf{a}^T\mathbf{b} &= a_1b_1 + a_2b_2 \\
&= ab\cos\alpha\cos\beta + ab\sin\alpha\sin\beta
\end{aligned}
$$

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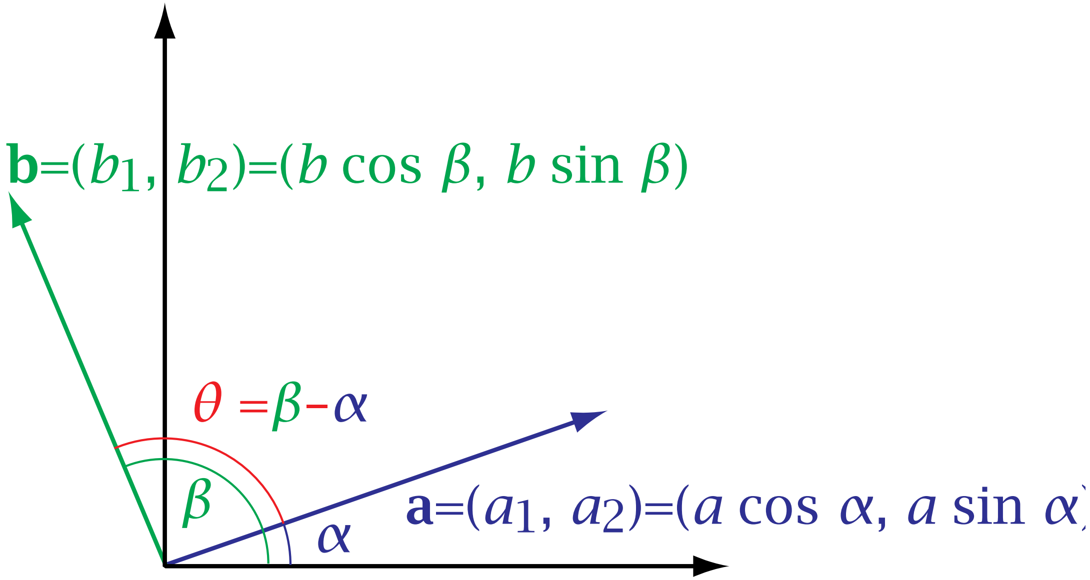
If **a** and **b** are two vectors that are separated by an angle θ, then their inner product can be alternatively defined as $\mathbf{a}^T\mathbf{b} = ab\cos\theta$



$$
\begin{aligned}
\mathbf{a}^T\mathbf{b} &= a_1b_1 + a_2b_2 \\
&= ab\cos\alpha\cos\beta + ab\sin\alpha\sin\beta \\
&= ab\cos(\beta-\alpha) = ab\cos\theta
\end{aligned}
$$

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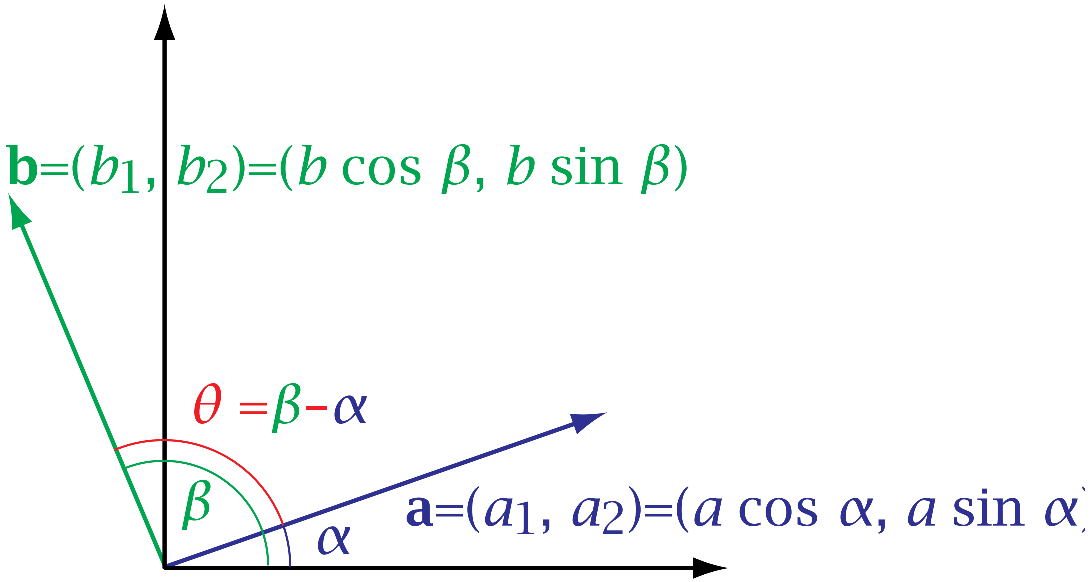

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&= ab\cos(\beta - \alpha) = ab\cos\theta
\end{aligned}
$$

(Note: this definition generalizes to vectors in higher dimensional spaces)

Outer Product

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^2$ their outer product is defined by

$$\mathbf{ab}^T = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

(Note: this definition generalizes to vectors in higher dimensional spaces)

Unit Vectors

Let $\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ denote the *unit vectors* along the two principal coordinate directions (x and y). Let $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ denote an arbitrary point in this coordinate system. Then we obtain the *inner products*

$$\hat{\mathbf{x}}^T \mathbf{p} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x.$$

Similarly,

$$\hat{\mathbf{y}}^T \mathbf{p} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = y.$$

What are the *outer products* $\hat{\mathbf{x}}\hat{\mathbf{x}}^T$? $\hat{\mathbf{y}}\hat{\mathbf{y}}^T$?

Identity Matrix (Review)

Observe that

$$I = \hat{\mathbf{x}}\hat{\mathbf{x}}^T + \hat{\mathbf{y}}\hat{\mathbf{y}}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ denote the unit vectors parallel to the two positive axes of a new coordinate system that appears θ radians counter-clockwise with respect to the principal directions.