Tom Cover's Number Guessing Game

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Rules

Alice generates two independent random numbers, X_1 and X_2 from a fixed, but hidden, continuous, probability distribution, $F(x) = \mathbb{P}\{X \leq x\}$.

After seeing X_1, Bob has to guess if either

1. $X_1 < X_2$, or
2. $X_1 \geq X_2$.

If Bob guesses correctly, he wins. If he guesses incorrectly he loses.

Can Bob win *more* than half of the time? And if so, *how?*

Strategy if Bob knows $F(x)$

Since F is the distribution for both X_1 and X_2,

$$F(X_1) = \mathbb{P}\{X_2 \leq X_1\}.$$

Thus, Bob could apply the following algorithm: Given X_1, compute $F(X_1)$.

1. If $F(X_1) < \frac{1}{2}$, then guess that $X_1 < X_2$,
2. Otherwise, guess that $X_1 \geq X_2$.

Claim 1: Using this algorithm, Bob is correct $\dfrac{3}{4}$ of the time.

What can Bob do if he doesn't know $F(x)$?

Strategy: Bob should generate a *new* random number Y, from his own distribution $G(y)$, with support over $\mathbb{R}$ (i.e., $a < b \Rightarrow G(a) < G(b)$).

Then apply the following algorithm:

1. If $X_1 < Y$ then guess that $X_1 < X_2$,
2. Otherwise, guess that $X_1 \geq X_2$.

Claim 2: Using this algorithm, Bob will guess correctly *more* than half of the time.

Continuous Random Variables

A *real-valued random variable* is a function $\mathbf{X} : \Omega \to \mathcal{S} \subset \mathbb{R}$,

A real-valued random variable is *measurable* if

$$\{\omega \in \Omega : \mathbf{X}(\omega) < x\} \in \mathcal{A}, \forall x \in \mathcal{S}.$$

We define the *probability distribution* of the real-valued, random variable $\mathbf{X}$, as

$$F_{\mathbf{X}}(x) = \mathbb{P}\{\mathbf{X} < x\} \triangleq \mathbb{P}\{\omega \in \Omega : \mathbf{X}(\omega) < x\},$$

where the subscript of $F_{\mathbf{X}}$ is often ommitted if no confusion arises.

Likewise, we define the *probability density* of $\mathbf{X} \in \mathcal{S}$, as

$$f_{\mathbf{X}}(x) = \frac{d}{dx} F_{\mathbf{X}}(x)$$

at all points $x \in \mathcal{S}$ where the derivative is defined.

Proof of Claim 1

$$\begin{aligned}\mathbb{P}\{X_2 > X_1 | X_1\} &= \int_{X_1}^{+\infty} f(x_2)\, dx_2 \\ &= \int_{X_1}^{+\infty} dF(x_2) \\ &= F(+\infty) - F(X_1) = 1 - F(X_1).\end{aligned}$$

$$\begin{aligned}\mathbb{P}\{X_2 < X_1 | X_1\} &= \int_{-\infty}^{X_1} f(x_2)\, dx_2 \\ &= \int_{-\infty}^{X_1} dF(x_2)\, dx_2 \\ &= F(X_1) - F(-\infty) = F(X_1) - 0 = F(X_1)\end{aligned}$$

Proof of Claim 1 (cont.)

$$\begin{aligned}
\mathbb{P}\{\text{Bob is correct}\} &= \mathbb{P}\left\{X_2 > X_1, F(X_1) < \frac{1}{2}\right\} + \mathbb{P}\left\{X_2 \leq X_1, F(X_1) \geq \frac{1}{2}\right\} \\
&= \int_{\{x_1 \in \mathbb{R} \mid F(x_1) < \frac{1}{2}\}} (1 - F(x_1))\, dF(x_1) \\
&\quad + \int_{\{x_1 \in \mathbb{R} \mid F(x_1) \geq \frac{1}{2}\}} F(x_1)\, dF(x_1) \\
&= F\left(1 - \frac{1}{2}F\right)\Bigg|_0^{\frac{1}{2}} + \frac{1}{2}F^2\Bigg|_{\frac{1}{2}}^1 \\
&= \frac{3}{4}.
\end{aligned}$$

Proof of Claim 2

$$\begin{aligned}
\mathbb{P}\{\text{Bob's correct}|Y\} &= \mathbb{P}\{X_2 \le X_1, Y < X_1|Y\} + \mathbb{P}\{X_2 > X_1, Y > X_1|Y\} \\
&= \int_Y^{+\infty} \mathbb{P}\{X_2 \le x_1\} f(x_1)\,dx_1 + \int_{-\infty}^{Y} \mathbb{P}\{X_2 > x_1\} f(x_1)\,dx_1 \\
&= \int_Y^{+\infty} F(x_1)\,f(x_1)\,dx_1 + \int_{-\infty}^{Y} (1 - F(x_1))\,f(x_1)\,dx_1 \\
&= \int_Y^{+\infty} F(x_1)\,dF(x_1) + \int_{-\infty}^{Y} (1 - F(x_1))\,dF(x_1) \\
&= \frac{1}{2}F(x_1)^2\Big|_Y^{+\infty} + F(x_1)\Big|_{-\infty}^{Y} - \frac{1}{2}F(x_1)^2\Big|_{-\infty}^{Y} \\
&= \frac{1}{2} + F(Y)\,(1 - F(Y)).
\end{aligned}$$

Proof of Claim 2 (cont.)

Since,

$$\mathbb{P}\{\text{Bob's correct}|Y\} = \frac{1}{2} + F(Y)\left(1 - F(Y)\right),$$

$$\begin{aligned}\mathbb{P}\{\text{Bob's correct}\} &= \int_{-\infty}^{+\infty} \mathbb{P}\{\text{Bob's correct}|Y = y\}g(y)\,dy \\ &= \frac{1}{2} + \int_{-\infty}^{+\infty} F(y)\left(1 - F(y)\right)dG(y) \\ &> \frac{1}{2}.\end{aligned}$$

References

1. T. Cover, “Pick the largest number,” in T. Cover and B. Gopinath, ed. *Open Problems in Communcations and Computations*, Springer-Verlag, New York, 1987, p. 152.
2. Peter Winkler, “Games People Don’t Play,” in David Wolfe and Tom Rodgers, ed., *Puzzlers’ Tribute: A Feast for the Mind*, A. K. Peters, Natick, MA, 2002, pp. 301–313.