1. Prove or refute for $j, n \in \mathbb{N}_0$ with $j \leq n$,

$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}.$$

2. Prove or refute for $n, x \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x \bmod n = 1 \Rightarrow x^k \bmod n = 1$.

3. Prove or refute for $n \in \mathbb{N}$, $x, y \in \mathbb{N}_0$, $(x \times y) \bmod n = ((x \bmod n) \times (y \bmod n)) \bmod n$.

4. Prove or refute for $n, y \in \mathbb{N}$, $x \in \mathbb{N}_0$, $(x/y) \bmod n = ((x \bmod n)/(y \bmod n)) \bmod n$.

5. Prove or refute, for $n \in \mathbb{N}$, $i, j \in \mathbb{N}_0$, and $m \in \{0, 1, 2, \ldots, n-1\}$, then

$$m^{i+j} \bmod n = \big((m^i \bmod n) \cdot (m^j \bmod n)\big) \bmod n$$

6. Prove, for $x \in \mathbb{R}$, the *indispensable* formula

$$\sum_{k=0}^{n} x^k = \begin{cases} n+1, & \text{if } x = 1, \\ \dfrac{x^{n+1}-1}{x-1}, & \text{otherwise.} \end{cases}$$

   (Letting $x = 2$, the above evaluates to the total number of nodes in a symmetric binary tree of depth n, among other things.) Then, for the case $|x| < 1$, evaluate the limit,

$$\sum_{k=0}^{\infty} x^k = \lim_{n \to \infty} \sum_{k=0}^{n} x^k = ?$$

   (The above is called a *geometric series*.)

7. Let m, n, q, and r denote any four natural numbers that satisfy, $m = qn + r$; hence, $q = \lfloor m/n \rfloor$ and $r = m \bmod n$. Prove that

$$\gcd(m, n) = \gcd(n, r).$$

   The above theorem is the basis for Euclid's recursive algorithm, here written in pseudocode:

```
int gcd(int m, int n) {
  if (m < n) {         // swap m and n
    int t ← m;
    m ← n;
    n ← t;}
  int r ← m mod n;     // compute the remainder
  if (r ≡ 0) {
     return n;         // Eureka!
  } else {
     return gcd(n, r); // recursion
  }}
```

8. Apply Euclid's recursive algorithm to find the greatest common divisor of 77639912 and 4504864. Show each step of your computation. (You may use a calculator to perform your divisions.)

9. The Fibonacci numbers F_n are defined by the recursion relation,

$$F_{n+2} = F_{n+1} + F_n, \text{ for } n \in \mathbb{N}_0,$$

with $F_0 = 0$ and $F_1 = 1$. How many applications (or iterations) of Euclid's algorithm are required to evaluate $\gcd(F_{k+1}, F_k)$ for an arbitrary $k \in \mathbb{N}_0$. What is the ultimate answer? *Extra credit* for answering the above for $\gcd(F_j, F_k)$, for arbitrary $j, k \in \mathbb{N}_0$.

10. The Towers of Hanoi puzzle is described in among other places on pages II:DT:18–20. Let $M(n)$ denote the minimum number of moves required to transfer $n \in \mathbb{N}$ disks from the initial pole to the goal pole for a puzzle with three poles. Show that $M(n)$ satisfies the recursion relation

$$M(n+1) = 2M(n) + 1,$$

with $M(1) = 1$. Use induction to prove that

$$M(n) = 2^n - 1.$$

How many years are needed to solve the puzzle with $n = 64$ disks, if one can perform exactly one move per second? Compare this number with scientific estimates of the age of the universe.