Set Theory

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1 Basic Definitions

A *set* is defined as collection of objects, or elements. Familiar examples are

$\mathbb{B} \triangleq \{0, 1\}$,	the set of *binary values*
$\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$,	the set of *natural numbers*
$\mathbb{N}_0 \triangleq \{0, 1, 2, \ldots\}$,	the set of *non-negative integers*
$\mathbb{Z} \triangleq \{\ldots, -1, 0, 1, \ldots\}$,	the set of *integers*
$\mathbb{Z}_- \triangleq \{-1, -2, -3, \ldots\}$,	the set of *negative integers*
$\mathbb{Q}$	the set of *rational numbers*
$\mathbb{R}$	the set of *real numbers*
$\mathbb{C}$	the set of *complex numbers*

Here, *ellipses* (the symbol "...") can represent either a finite or infinite number of implied elements. We will often use capital Roman letters, such as A, B, C to denote different sets, and lowercase letters (x, y, z) to denote elements. When defining sets using the brace notation, e.g., $\{x, y, z\}$, it is important to adhere to the rule that each element should only appear once.

For every system of sets, there are two special sets. The set of all elements is called the *universal set*, and is denoted in general by Ω. The universal set thus defines the context from which other sets can be formed. For example, for systems of sets consisting of only real numbers, the universal set would be the set of real numbers, $\mathbb{R}$. For subsets that involve a standard deck of playing cards, we let Ω

$$\Omega = \{A\clubsuit, 2\clubsuit, \ldots, K\clubsuit, A\diamondsuit, 2\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, 2\heartsuit, \ldots, K\heartsuit, A\spadesuit, 2\spadesuit, \ldots, K\spadesuit\}.$$

The second special set is the set without any members, which called the *empty set*, designated by $\emptyset$, or by $\{\}$. $\emptyset$

If an element x belongs to a set A, we then state that x *is a member of* A. More concisely, we write $x \in A$. (This is another example of a logical predicate of two arguments.) If x is not a member of A, we write $\overline{x \in A}$, $\neg(x \in A)$, or more concisely, $x \notin A$. For example, $\pi \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Z}$, are true propositions. To simplify notation, we will write $x, y \in A$ to mean that x and y are both elements of set A, instead of $x \in A \wedge y \in A$. $\in, \notin$

Frequently, we will define a set as the largest collection of elements that satisfies a particular logical statement. Such a set is called a *truth set*. By this method we use the notation $\{e(x, y, \ldots) : S(x, y, \ldots)\}$, where $x, y, \ldots$ represent an arbitrary sequence of variable arguments, $e(\cdot)$, an expression of those arguments, and $S(\cdot)$, a logical sentence. This set is specified by the entire collection of values, $e(x, y, \ldots)$ that correspond to every value of $x, y, \ldots$ that makes the sentence $S(x, y, \ldots)$ true. (Frequently, the colon delimiter is replaced by a vertical bar.) For example, the set of *rational numbers* is defined as

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z} \wedge n \neq 0 \right\}.$$

In the event that $e(\cdot)$ above represents a single variable, it is often clearer to move the membership portion of the sentence before the colon. Thus,

$$\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}.$$

If every member of A is also a member of B, we say that A is a *subset* of B, and write $A \subseteq B$ (or $B \supseteq A$). If A is *not a subset* of B, we write $A \not\subseteq B$. If every member of A belongs to B, and B contains at least one member that does not belong to A, we say that A is a *proper subset* of B, and write $A \subset B$ (or $B \supset A$). Thus, $\mathbb{N} \subset \mathbb{Z}$ because all natural numbers are also integers, but not all integers are natural numbers. $\subseteq$ $\subset$

Two sets A and B are said to be equal if the contain exactly the same elements, or alternatively if $(A \subseteq B) \wedge (B \subseteq A)$. In this case we write $A = B$.

Note that the set relations $\subseteq$, $\subset$, and $=$ are *transitive*: e.g., $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$.

On occasion we may find it useful to consider sets that have other sets as members. In such cases a set of sets is often called a *class* or *family*, and a set of classes or families, a *collection*. An example of a class is the *power set* of A, denoted by $\mathcal{P}(A)$, which is defined as the set of all subsets of A:

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Thus, for example $\mathcal{P}(\mathbb{B}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

For $m, n \in \mathbb{Z}$, with $m \leq n$, we will use the notation

$$\{m \ldots n\} \triangleq \{i \in \mathbb{Z} : m \leq i \leq n\},$$

to indicate a simple discrete interval, i.e., the set of all integers that range from m to n. Likewise, we define intervals over the reals in four different ways:

$$\begin{aligned}
(a, b) &\triangleq \{x \in \mathbb{R} : a < x < b\} \\
(a, b] &\triangleq \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &\triangleq \{x \in \mathbb{R} : a \leq x < b\} \\
[a, b] &\triangleq \{x \in \mathbb{R} : a \leq x \leq b\},
\end{aligned} \tag{1}$$

for any a and b that satisfy $-\infty < a \leq b < +\infty$. To these we add the infinite interval $(-\infty, +\infty) = \mathbb{R}$, and the semi-infinite intervals $(-\infty, b]$ and $[a, +\infty)$, for any finite a and b. Likewise, for $a \in \mathbb{R}$, $(a, a] = [a, a) = [a, a] = \{a\}$, and $(a, a) = \emptyset$.

The *cardinality* of a set A, denoted by $|A|$, is defined to be the number of members it contains. Thus, $|\emptyset| = 0$, $|\mathbb{B}| = 2$ and $|\mathbb{N}| = \infty$.

A set A is said to be *finite* if $|A| < \infty$. A set is said to be *countable* if its elements can be placed into a one-to-one correspondence with a subset of the natural numbers. A set is said to be *countably infinite* if its elements can be placed into one-to-one correspondence with the natural numbers.

EXERCISE 1. Show that the sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

> *Extra for experts:* The German mathematician Georg Cantor (1845–1918) created quite a stir by showing that it was not possible to construct a one-to-one correspondence between the natural numbers, and a bounded interval of real numbers, e.g., the unit interval $[0, 1]$. In fact, there are many more numbers in the unit real interval, than there are in the set $\mathbb{N}$[1]. He thus demonstrated that it is incorrect to assume that there is only one infinity, ∞. Rather there is an infinite hierarchy of infinities, $\aleph_0 < \aleph_1 < \aleph_2 < \cdots$, called "aleph null," "aleph one," etc. This result is more than esoteric. It forces us to treat real-valued random variables with special care in order to avoid absurd conclusions.
>
> Cantor also showed that the real numbers are not countable by means of an elegant diagonalization technique and contradiction. Let's assume that the numbers in the real interval can be placed into a one-to-one correspondence with the naturals. In the table below, naturals appear on the left, the reals in [0, 1] on the right. We represent each number on the right by its decimal expansion, which is non-repeating for the transcendental reals, such as $\pi - 3$. Also note that we don't care about placing the reals in any particular order.
>
> Now if our assumption is correct, every real number in the unit interval appears somewhere in the right column. However, by changing the value of each diagonal digit (underlined), we produce a number that cannot appear in the right column, as it differs from every entry therein. For example, let z denote a particular number that disagrees with the diagonal digit of every number above, i.e. its first digit is different

[1] His method used a powerful diagonalization technique, that eventually lead to the even greater stir in logic caused by Kurt Gödel.

$\mathbb{N}$	$[0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$
1	$0.\underline{6}1250987362839402938420\cdots$
2	$0.9\underline{8}983749283497383223384\cdots$
3	$0.73\underline{3}97403982739483333049\cdots$
4	$0.141\underline{5}9265358979323846264\cdots$
5	$0.2500\underline{0}000000000000000000\cdots$
6	$0.12345\underline{6}7890123456789 0123\cdots$
$\vdots$	$\vdots$

from 36, its second is different from 8, its third different from 3, etc. Let's choose, $z = 0.194617\cdots$. But if I were now to claim that z actually appears somewhere in the table, say on the 145,763,804th row, you could rightly say, "that's impossible, because the 145,763,804th digit of z differs from the number that actually appears in that row." Thus the real numbers in the unit interval cannot be placed into a one-to-one correspondence with the integers, and therefore the real numbers are not countable.

1.1 Set Operations

The standard operations that produce a new set from one or more given sets, are based on the Boolean unary and binary operators. Using Boolean negation, we define the *complement* of a set A as the set of all elements that do not belong to A. Here, we denote the complement of A by the notation A^c, whence A^c, ^{c}

$$A^c = \{x \in \Omega : x \notin A\}.$$

Note that this operation requires that the universal set Ω be defined. From the definitions of Ω and $\emptyset$ it follows that $\Omega^c = \emptyset$, and $\emptyset^c = \Omega$.

Using Boolean disjunction, we define the *union* of A and B, as the set of elements that belong to at least one of these two sets. This union is denoted as $A \cup B$, whence, $\cup$

$$A \cup B = \{x : x \in A \vee x \in B\}.$$

For example,

$$\{1,2,3,4\} \cup \{3,4,5,6\} = \{1,2,3,4,5,6\}.$$

Likewise, using Boolean conjunction, we define the *intersection* of A and B, as the set of elements that belong to both of these two sets. This intersection is denoted as $A \cap B$, whence, $\cap$

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

Thus, for example,

$$\{1,2,3,4\} \cap \{3,4,5,6\} = \{3,4\}.$$

Two arbitrary sets, A and B are said to be *disjoint* if $A \cap B = \emptyset$.

The *difference* between sets A and B, denoted by $A \setminus B$, consists of the members of A that do not belong to B. Thus, $\setminus$

$$A \setminus B = \{x : x \in A \wedge x \notin B\}.$$

For example,

$$\{1,2,3,4\} \setminus \{3,4,5,6\} = \{1,2\},$$

and

$$\{3,4,5,6\} \setminus \{1,2,3,4\} = \{5,6\},$$

Note that $A \setminus B = A \cap B^c$.

The *symmetric difference* between A and B, denoted by $A \triangle B$, consists of the elements that are unique to each set. Thus, $\triangle$

$$A \triangle B \triangleq (A \setminus B) \cup (B \setminus A)$$

whence,

$$\{1,2,3,4\} \triangle \{3,4,5,6\} = \{1,2,5,6\},$$

In the exercises below, the reader is asked to prove the following results, where A, B, C denote sets.

$$(A^c)^c = A \tag{2}$$
$$A \cap B = B \cap A \tag{3}$$
$$A \cap (B \cap C) = (A \cap B) \cap C \tag{4}$$
$$A \cup B = B \cup A \tag{5}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \tag{6}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{7}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{8}$$
$$(A \cap B)^c = A^c \cup B^c \tag{9}$$
$$(A \cup B)^c = A^c \cap B^c \tag{10}$$
$$A \subseteq B \Leftrightarrow B^c \subseteq A^c \tag{11}$$
$$A \subset B \wedge B \subset C \Rightarrow A \subset C \tag{12}$$

1.2 Venn Diagrams

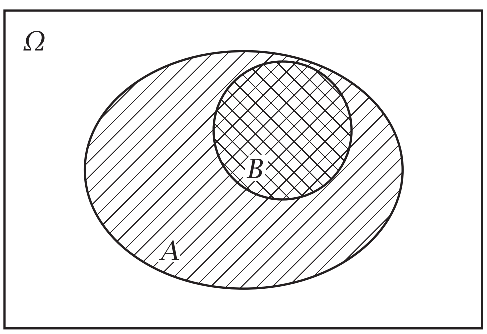

Figure 1: A Venn diagram of two sets, $B \subset A$.

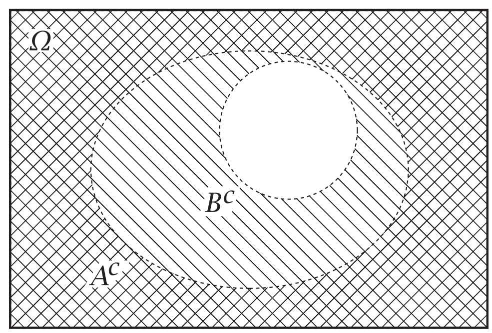

Figure 2: The complements of the previous two sets. The crosshatching evident in the region corresponding the A^c suggests that $B \subseteq A \Rightarrow A^c \subset B^c$.

The English logician, John Venn (1834–1923) introduced the use of two-dimensional diagrams to help visualize abstract set relations and operations (Figs. 1 through 3). In each of these diagrams is constructed in the context of an abstract *universal set*, Ω that is depicted as the enclosing rectangle. Each subset of Ω is represented by a set of points within the rectangle, usually by an elliptical region. In order to distinguish one subset from the other, their interiors are sometimes shaded distinctly. In Fig. 1, two intersecting subsets of Ω are shown, one as subset A (shaded with right-handed diagonals.[2]), and the other as subset B (with left-handed diagonals). Because B is a subset of A, the region interior to B exhibits both kinds of shading. This suggests the hypothesis, $B \subseteq A \Rightarrow A \cap B = B$. However, since Venn diagrams only illustrate one particular instance of a set theory problem, they cannot provide a general forward proof of a theorem. Rather, they can help illustrate the meaning of complex set theoretic expressions, and can be used to refute a proposition by depicting a carefully chosen counter example.

[2]Here, *right-handed diagonals* run from the upper right to the lower left. They are the diagonals that are easiest to draw with one's right hand. Conversely, *left-handed diagonals* run from the upper left to the lower right.

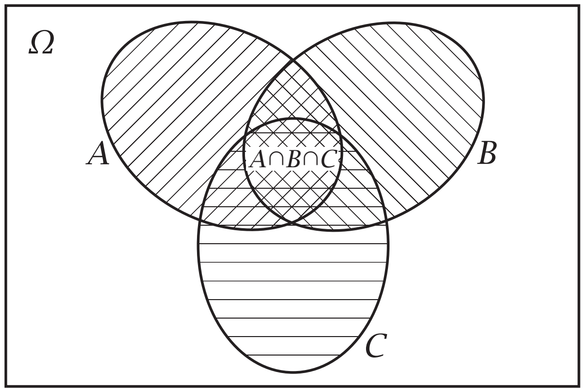

Figure 3: A Venn diagram of three intersecting sets A, B, and C. Set A is represented by right diagonals, set B by left diagonals, and set C by horizontal rules.

The following proof of the above hypothesis follows the "element method" as described in Bender and Williamson [2005, p. SF–4]:

> We show that $A \subseteq B$ implies *both* of the following statements: (i) $A \cap B \subseteq B$, and (ii) $B \subseteq A \cap B$. To demonstrate (i) we use the definition of set intersection: if $x \in A \cap B$, then $x \in A$ *and* $x \in B$. This, in turn, by the definition of subset, implies that $A \cap B$ is a subset of B (as well as of A). To demonstrate (ii) we begin with the definition of $B \subseteq A$, which implies $x \in B \Rightarrow x \in A$. Since now $x \in A \wedge x \in B$, it follows that $x \in A \cap B$. Consequently, by the definition of subset, $B \subset A \cap B$.

The complements of sets A and B in Fig. 1 are depicted in Fig. 2. Here, A^c is the region that is exterior to the ellipse that defines A, and is shaded with right-handed diagonals. Similarly, B^c is the region exterior to the circle that defines B, and is shaded with left-handed diagonals. That the region defining A^c is cross hatched suggests that $A^c \subset B^c$, and thus (perhaps) that $B \subset A \Leftrightarrow A^c \subset B^c$. (Can you prove this?)

1.3 Sequences of sets

Given a sequence of sets $A_1, A_2, \ldots, A_n$, and for $1 \le r \le s \le n$, we define the finite union,

$$\bigcup_{i=r}^{s} A_i \triangleq A_r \cup A_{r+1} \cup \cdots \cup A_s,$$

and the finite intersection,

$$\bigcap_{i=r}^{s} A_i \triangleq A_r \cap A_{r+1} \cap \cdots \cap A_s.$$

In the event that $r > s$ we adopt the conventions,

$$\bigcup_{i=r}^{s} A_i = \emptyset, \quad \text{and,} \quad \bigcap_{i=r}^{s} A_i = \Omega.$$

Using induction, one can generalize many of the set identities defined above so that they involve countable unions or intersections. For example, De Morgan's Laws generalize to

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} {A_i}^c, \quad \text{and,} \quad \left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=n}^{n} {A_i}^c$$

Letting,

$$\bigcup_{i=r}^{\infty} A_i \triangleq \lim_{s\to\infty} \bigcup_{i=r}^{s} A_i, \quad \text{and,} \quad \bigcap_{i=r}^{\infty} A_i \triangleq \lim_{s\to\infty} \bigcap_{i=r}^{s} A_i.$$

we obtain,

$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} {A_i}^c, \quad \text{and,} \quad \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=n}^{\infty} {A_i}^c$$

A countable family of sets $\{A_1, A_2, \ldots\}$ is said to be *disjoint* if $A_i \cap A_j = \emptyset$ whenever $i \neq j$. A countable family is said to be *complete* if $\bigcup_n A_n = \Omega$. A family of sets that is both disjoint and complete is called a *partition* of Ω.

1.4 Problems

1.1 Evaluate $(\{1, 2, 3\} \cup \{2, 3, 4\}) \cap (\{1, 2, 3\} \setminus \{3, 4, 5\})$.

1.2 Evaluate the power set of

- (a) $\{0, 1, 2\}$,
- (b) $\{\emptyset, \Omega\}$
- (c) $\mathscr{P}(\{\emptyset, \Omega\})$.

1.3 Create a copy of the Venn diagram that appears in Fig. 3, and label each of the 8 homogeneously shaded region in terms of A, B, and C. For example, the central region that contains all three shadings (left diagonal, right diagonal, and horizontal rules) is $A \cap B \cap C$.

1.4 Prove each of the following hypotheses using the "element method:"

- (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
- (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- (c) $A \subset B$ and $B \subset C \Rightarrow A \subset C$, (*transitivity of subset*).
- (d) $A \triangle B = (A \cup B) \setminus (A \cap B)$.
- (e) $\emptyset \subseteq A$ for every set A.
- (f) $(A \cap B) \subseteq (A \cup B)$ for any sets A and B.
- (g) $(A^c)^c = A$.

1.5 Use the "tabular method" [Bender and Williamson, 2005, p. SF–4] to show that

$$A \triangle B = \big(A \cup B\big) \cap \big(A^c \cup B^c\big).$$

1.6 Use the "algebraic method" [Bender and Williamson, 2005, p. SF–5] to show

- (a) $A \cup B = A \cup (B \setminus (A \cap B)$,
- (b) $A \cup B \cup C = A \cup (B \setminus A) \cup \big(C \setminus (A \cup B)\big)$.

1.7 Show that $A \subset B \Rightarrow |A| \leq |B|$.

1.8 Show that $\Omega^c = \emptyset$ and $\emptyset^c = \Omega$.

1.9 Show that $A \cap \Omega = A$ and $A \cup \emptyset = A$.

1.10 Show that $A \cap (B \setminus (A \cap B)) = \emptyset$.

References

E. A. Bender and S. G. Williamson. *A Short Course in Discrete Mathematics*. Dover, Mineola, NY, 2005.

