Math 331 Final Exam Solutions **Spring, 2010**

The five numbered problems are worth 20 points each, for a total of 100. The two extra credit problems are worth 30 points each. You may use without proof any facts from freshman calculus or Math 241 (e.g., L'Hospital's Rule, the values of familiar limits and integrals). The symbol D will always denote the unit disk: $D = \{z \in \mathbf{C} : \ |z| < 1\}$. The symbol $\mathcal{H}$ will always denote the right half-plane: $\mathcal{H} = \{z : \ \Re(z) > 0\}$. If $\Omega \subset \mathbf{C}$ is any open set, $H(\Omega)$ will denote the family of analytic functions defined on Ω.

1. Show that, if $a > 0$,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \pi e^{-a}.$$

Solution. Define $f(z) = ze^{iz}/(z^2 + a^2)$. For any $R > 0$,

$$\int_{-R}^{R} f(x)\, dx = i \int_{-R}^{R} \frac{x \sin x}{x^2 + a^2}\, dx,$$

which converges to i times our integral as $R \to \infty$. For $R > a$, let γ_R be the path that runs along $\mathbf{R}$ from $-R$ to R, then counterclockwise along a circular arc from R back to $-R$. The function f has a simple pole at $z = ia$, with residue

$$\frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.$$

Therefore

$$\int_{\gamma_R} f(z)\, dz = 2\pi i \frac{e^{-a}}{2} = i\pi e^{-a}.$$

Denote the curved part of γ_R by C_R. It now suffices to show that

$$\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0.$$

If we parameterize C_R as $C_R(s) = Re^{is}$, for $0 \le s \le \pi$, the absolute value of the integral is seen to be less than or equal to

$$\int_0^{\pi} \frac{R^2}{R^2 - a^2} e^{-R \sin s}\, ds \le 2 \int_0^{\pi} e^{-R \sin s}\, ds,$$

when $R >> a$. But, as we know from homework, the last expression goes to 0 as $R \to \infty$. Done.

2. Show that, if $a > 0$,

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2}\, dx = \frac{\pi}{2a} \log a.$$

Solution. For $0 < \epsilon << a << R$, let $\gamma_{\epsilon,R}$ be the path that runs straight from $-R$ to $-\epsilon$, clockwise on a circular arc from $-\epsilon$ to ϵ, straight from ϵ to R, and then counterclockwise on a circular arc from R to $-R$. Let $f(z) = \frac{\log z}{z^2+a^2}$, where we choose the branch of $\log z$ defined on the slit plane $\mathbf{C} \setminus \{it : \ t \le 0\}$, and such that $\log 1 = 0$. The function f has a simple pole at $z = ia$ with residue

$$\frac{\log(ia)}{2ia} = \frac{\log a + i\pi/2}{2ia} = -i\frac{\log a}{2a} + \frac{\pi}{4a}.$$

The integral of f around $\gamma_{\epsilon,R}$ equals

$$2\int_\epsilon^R \frac{\log x}{x^2+a^2}\, dx + i\pi \int_\epsilon^R \frac{dx}{x^2+a^2} + (I)_\epsilon + (II)_R,$$

where $(I)_\epsilon$ and $(II)_R$ denote the integrals over the "curvy" parts (of respective radii ϵ and R). For small ϵ, the magnitude of $(I)_\epsilon$ is no bigger than

$$\frac{\pi\epsilon(\log(1/\epsilon)+\pi)}{a^2-\epsilon^2} \to 0$$

as $\epsilon \to 0$. Similarly, the magnitude of $(II)_R$ is less than or equal to

$$\frac{\pi R(\log(R)+\pi)}{R^2-a^2} \to 0$$

as $R \to \infty$. Therefore, letting $\epsilon \to 0$ and $R \to \infty$, we get:

$$2\int_0^\infty \frac{\log x}{x^2+a^2}\, dx + i\pi \int_0^\infty \frac{dx}{x^2+a^2} = 2\pi i\left(-i\frac{\log a}{2a} + \frac{\pi}{4a}\right) = \frac{\pi \log a}{a} + \frac{\pi^2 i}{2a}.$$

Equating real parts and dividing by 2, we get

$$\int_0^\infty \frac{\log x}{x^2+a^2}\, dx = \frac{\pi \log a}{2a},$$

as desired.

3. Let $\mathcal{F} = \{f \in H(D) : \ \sup_{z\in D} |f(z)| \le 1 \text{ and } f(0) = 1/2\}$, and define

$$L_1 \equiv \inf_{f\in\mathcal{F}} \int_{-1/2}^{1/2} |f(t)|^2\, dt$$

$$L_2 \equiv \sup_{f\in\mathcal{F}} \int_{-1/2}^{1/2} |f(t)|^2\, dt.$$

Show that $L_1 > 0$ and $L_2 < 1$. (Hint: First show that there exist functions f_1 and f_2 in $\mathcal{F}$ such that

$$L_1 = \int_{-1/2}^{1/2} |f_1(t)|^2 \, dt$$

and

$$L_2 = \int_{-1/2}^{1/2} |f_2(t)|^2 \, dt.)$$

Solutions. Let $\{g_n\} \subset \mathcal{F}$ be a sequence such that

$$\int_{-1/2}^{1/2} |g_n(t)|^2 \, dt \to L_1.$$

Since $\mathcal{F}$ is locally bounded, we can pick a subsequence (which we shall also call $\{g_n\}$) that converges uniformly on compact subsets of D to some $g \in H(D)$. It's trivial that $g \in \mathcal{F}$. The uniform convergence of the g_n's implies that

$$\int_{-1/2}^{1/2} |g(t)|^2 \, dt = \lim_n \int_{-1/2}^{1/2} |g_n(t)|^2 \, dt = L_1.$$

If $L_1 = 0$ then $g = 0$ on all of $[-1/2, 1/2]$, which contradicts $g(0) = 1/2$. An almost identical argument shows that if $L_2 = 1$ then there is a $g \in \mathcal{F}$ such that $\int_{-1/2}^{1/2} |g(t)|^2 \, dt = 1$, implying that $|g| \equiv 1$ on $[-1/2, 1/2]$, which also contradicts $g(0) = 1/2$.

4. Let $\{f_n\}_1^\infty$ be a sequence of functions in $H(D)$. For each n define $M_n \equiv \sup_{z \in D} |f_n(z)|$, and suppose that $\sum_n M_n < \infty$. Show that the infinite series

$$\sum_n f_n'(z)$$

converges for all $z \in D$.

Solution. The partial sums

$$\sum_{k=1}^n f_k(z)$$

converge uniformly on all compact subsets of D. Therefore their derivatives,

$$\sum_{k=1}^n f_k'(z),$$

do too. End of story.

5. Define $\Omega = D \setminus \overline{\Delta(1/2; 1/2)}$. (I suggest you draw Ω!) Find a function $f \in H(\Omega)$ that maps Ω one-to-one and onto $\mathcal{H}$. (Hint: Begin by mapping all of D onto $\mathcal{H}$, and see where it takes Ω.)

Solution. Follow the hint. The map $z \mapsto (1+z)/(1-z)$ sends Ω to the strip $S \equiv \{z : 0 < \mathrm{Re}(z) < 1\}$. The map $z \mapsto e^{i\pi z}$ sends S to the upper half plane $U \equiv \{z : \mathrm{Im}(z) > 0\}$. The map $z \mapsto -iz$ sends U to $\mathcal{H}$.

Extra Credit 1. Using the notation from problem #3, show that $L_1 < L_2$.

Solution. Trivially, $L_1 \leq 1/4$ (just put $f \equiv 1/2 \in \mathcal{F}$). The function $(1+z)/2$ also belongs to $\mathcal{F}$. Let's work out

$$\int_{-1/2}^{1/2} \left(\frac{1+x}{2}\right)^2 dx.$$

The antiderivative is $(1/12)(1+x)^3$, so the integral is

$$(1/12)\left((3/2)^3 - (1/2)^3\right) = 13/48 > 1/4.$$

Therefore $L_2 > 1/4 \geq L_1$.

Extra Credit 2. Show—by exhibiting one—that there exists a sequence of functions in $H(D)$ satisfying the hypotheses of problem #4, but for which

$$\lim_{\substack{x \to 1^- \\ x \in \mathbf{R}}} \left| \sum_n f_n'(x) \right| = \infty.$$

Solution. Put $f_n(z) = z^{n^2}/n^2$ for $n \geq 1$. Then $f_n'(z) = z^{n^2-1}$. All of these are positive if $0 < x < 1$. For any N, there is a number $0 < r < 1$ such that $r < x < 1$ implies that all of $x^3, x^{15}, \ldots, x^{N^2-1}$ are $\geq 1/2$. Therefore, if $r < x < 1$,

$$\sum_1^\infty f_n'(x) \geq \sum_1^N x^{n^2-1} \geq \left(\sum_1^N (1/2)\right) \geq N/2,$$

which goes to infinity as $N \to \infty$.