Math 331 Homework #1 **Spring, 2010**

Assigned January 27. Due February 5. The problems are worth 14 points each, with 2 points added to make 100.

1. Let (X, d) be a metric space. Recall that a sequence $\{x_n\} \subset X$ is said to converge to $p \in X$ (written $x_n \to p$) if $d(x_n, p) \to 0$ as $n \to \infty$. (Formally: for every $\epsilon > 0$ there is an N such that, if $n \geq N$, then $d(x_n, p) < \epsilon$.) Show that $x_n \to p$ if and only if, for all open sets U containing p, there is an N such that, if $n \geq N$, then $x_n \in U$.

2. Let X be a non-empty set with two metrics, d_1 and d_2, and suppose that the metric spaces (X, d_1) and (X, d_2) have the same open sets: $U \subset X$ is open in (X, d_1) if and only if it's open in (X, d_2). Show that they also have the same convergent sequences. I.e., show that if $\{x_n\} \subset X$ is any sequence and $p \in X$ is any point, then $d_1(x_n, p) \to 0$ as $n \to \infty$ if and only if $d_2(x_n, p) \to 0$ as $n \to \infty$.

3. Prove the converse of #2: Show that if (X, d_1) and (X, d_2) have the same convergent sequences (in the sense of problem #2), then they have the same open sets (also in the sense of problem #2).

4. Let (X, d) be a metric space and define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that (X, ρ) is also a metric space. (The only challenge here will be proving that ρ satisfies the triangle inequality.)

5. Let (X, d) be a metric space, and let (X, ρ) be as in problem #4. Show that $\rho(x, y) < 1$ for all x and y in X, but that (X, d) and (X, ρ) *still have the same open sets* in the sense of problem #2.

6. Let (X, d) be a metric space. Recall that a sequence $\{x_n\} \subset X$ is called Cauchy if, for every $\epsilon > 0$, there is a number N such that, if m and n are $\geq N$, then $d(x_n, x_m) < \epsilon$. A *subsequence* of $\{x_n\}$ is a sequence of the form $\{x_{n_k}\}$, where $n_1 < n_2 < n_3 \cdots$. Show: If $\{x_n\} \subset X$ is a Cauchy sequence and some subsequence $\{x_{n_k}\}$ converges to $p \in X$, then $x_n \to p$. (I call this the "Sheep Going Over the Cliff" Lemma.)

7. Let (X, d) be a metric space and suppose that $A \subset X$ is connected. Show that if

$$A \subset B \subset \overline{A}$$

then B is connected.