Math 331 Homework #1 Solution Set **Spring, 2010**

Assigned January 27. Due February 5. The problems are worth 14 points each, with 2 points added to make 100.

1. Let (X, d) be a metric space. Recall that a sequence $\{x_n\} \subset X$ is said to converge to $p \in X$ (written $x_n \to p$) if $d(x_n, p) \to 0$ as $n \to \infty$. (Formally: for every $\epsilon > 0$ there is an N such that, if $n \geq N$, then $d(x_n, p) < \epsilon$.) Show that $x_n \to p$ if and only if, for all open sets U containing p, there is an N such that, if $n \geq N$, then $x_n \in U$.

Solution. Suppose $x_n \to p$ and $p \in U$, where U is open. Pick $\epsilon > 0$ such that $B(p; \epsilon) \subset U$ and N such that $n \geq N$ implies $d(x_n, p) < \epsilon$. For the other direction, note that $B(p; \epsilon)$ is an open set containing p.

2. Let X be a non-empty set with two metrics, d_1 and d_2, and suppose that the metric spaces (X, d_1) and (X, d_2) have the same open sets: $U \subset X$ is open in (X, d_1) if and only if it's open in (X, d_2). Show that they also have the same convergent sequences. I.e., show that if $\{x_n\} \subset X$ is any sequence and $p \in X$ is any point, then $d_1(x_n, p) \to 0$ as $n \to \infty$ if and only if $d_2(x_n, p) \to 0$ as $n \to \infty$.

Solution. Suppose that $d_1(x_n, p) \to 0$. We wish to show that $d_2(x_n, p) \to 0$. Let $U \subset X$ be open in (X, d_2), and suppose that $p \in U$. By hypothesis, U is also open in (X, d_1). Therefore (by problem #1), $x_n \in U$ for sufficiently large n. But then (again by problem #1), $d_2(x_n, p) \to 0$. The reverse implication comes from symmetry.

3. Prove the converse of #2: Show that if (X, d_1) and (X, d_2) have the same convergent sequences (in the sense of problem #2), then they have the same open sets (also in the sense of problem #2).

Solution. Let U be open in (X, d_1) and suppose, by contradiction, that U is *not* open in (X, d_2). Then there exists $p \in U$ such that $B_2(p; r) \not\subset U$ for any $r > 0$; and, for each $n = 1, 2, 3, \ldots$, we can find a point $x_n \in B_2(p; 1/n)$ such that $x_n \notin U$. Therefore $d_2(x_n, p) < 1/n \to 0$, implying $d_1(x_n, p) \to 0$ too. The result of problem #1 now implies $x_n \in U$ for sufficiently large n, which contradicts how we chose x_n. Therefore U is open in (X, d_2). The reverse implication comes from symmetry.

4. Let (X, d) be a metric space and define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that (X, ρ) is also a metric space. (The only challenge here will be proving that ρ satisfies the triangle inequality.)

Solution. The symmetry, non-degeneracy, and non-negativity of ρ are trivial. For the triangle inequality, it's enough to show this lemma: If a, b, and c are non-negative numbers such that $a \leq b + c$, then

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}.$$

Since $x/(1+x) \nearrow$ on $[0,\infty)$ the lemma is vacuous if $a \le b$ or $a \le c$. Therefore we can assume $a > b$ and $a > c$. But then:

$$\begin{aligned}\frac{a}{1+a} &\le \frac{b+c}{1+a} \\ &= \frac{b}{1+a} + \frac{c}{1+a} \\ &\le \frac{b}{1+b} + \frac{c}{1+c},\end{aligned}$$

proving the lemma. The triangle inequality follows by setting $a = d(x,z)$, $b = d(x,y)$, $c = d(y,z)$.

5. Let (X,d) be a metric space, and let (X,ρ) be as in problem #4. Show that $\rho(x,y) < 1$ for all x and y in X, but that (X,d) and (X,ρ) *still have the same open sets* in the sense of problem #2.

Solution. The inequality $\rho < 1$ is trivial. We note that, if $d(x,y) < 1$, then

$$\rho(x,y) \le d(x,y) \le 2\rho(x,y);$$

because, if $0 \le x < 1$, then

$$\frac{x}{1+x} \le x \le \frac{2x}{1+x}.$$

Conversely (and for a similar reason), if $\rho(x,y) < 1/2$, then

$$d(x,y) \le 2\rho(x,y) \le 2d(x,y).$$

Therefore, if $\{x_n\}$ is any sequence in X and $p \in X$, $d(x_n,p) \to 0$ if and only if $\rho(x_n,p) \to 0$. Now apply the result of problem #3.

6. Let (X,d) be a metric space. Recall that a sequence $\{x_n\} \subset X$ is called Cauchy if, for every $\epsilon > 0$, there is a number N such that, if m and n are $\ge N$, then $d(x_n,x_m) < \epsilon$. A *subsequence* of $\{x_n\}$ is a sequence of the form $\{x_{n_k}\}$, where $n_1 < n_2 < n_3 \cdots$. Show: If $\{x_n\} \subset X$ is a Cauchy sequence and some subsequence $\{x_{n_k}\}$ converges to $p \in X$, then $x_n \to p$. (I call this the "Sheep Going Over the Cliff" Lemma.)

Solution. Let $\epsilon > 0$. Pick N such that, if m and n are $\ge N$, then $d(x_n,x_m) < \epsilon/2$. Pick J so large that $n_J \ge N$ AND $d(x_{n_J},p) < \epsilon/2$. Define $\tilde{N} = \max(N,n_J)$. If $n \ge \tilde{N}$ then, since n and n_J are both $\ge N$,

$$d(x_n,p) \le d(x_n,x_{n_J}) + d(x_{n_J},p) < \epsilon.$$

7. Let (X,d) be a metric space and suppose that $A \subset X$ is connected. Show that if

$$A \subset B \subset \overline{A}$$

then B is connected.

Solution. Suppose, by contradiction, that B is disconnected. Let (U, V) be a disconnection for it. Either $A \cap U$ or $A \cap V$ is empty (otherwise (U, V) would disconnect A). Suppose $A \cap U = \emptyset$. Then $A \subset X \setminus U$. But $X \setminus U$ is a closed set. Therefore $\overline{A} \subset X \setminus U$, implying $\overline{A} \cap U = \emptyset$. But then $B \cap U \subset \overline{A} \cap U = \emptyset$, implying that (U, V) does not disconnect B. QED.