Math 331 Homework #3 Solutions **Spring, 2010**

Assigned February 24. Due March 15. The ten numbered problems are worth 20 points each, for a total of 200.

1. Recall that, on $\Omega \equiv \mathbf{C} \setminus (-\infty, 0]$, the analytic logarithm $\log z$ can be defined as $\ln|z| + i\theta(z)$, where $\theta(z)$ is the unique angle θ such that $-\pi < \theta < \pi$ and $z = |z|e^{i\theta}$. If $z = x + iy \in \Omega$, we can—with a slight abuse of notation—write $\log z = u(x, y) + iv(x, y)$, where $u(x, y) = \ln|z|$ and $v(x, y) = \theta(z)$. Show directly that u and v satisfy the Cauchy-Riemann equations in Ω. This will mean finding a formula (perhaps piece-wise defined) for $\theta(z)$.

Solutions. $u(x, y) = (1/2)\ln(x^2 + y^2)$, implying

$$
\begin{aligned}
u_x &= \frac{x}{x^2+y^2}, \\
u_y &= \frac{y}{x^2+y^2}.
\end{aligned}
$$

When $x > 0$, $v(x, y) = \arctan(y/x)$, and

$$
\begin{aligned}
v_x &= -(y/x^2)(1 + (y/x)^2)^{-1} = \frac{-y}{x^2+y^2} = -u_y, \\
v_y &= (1/x)(1 + (y/x)^2)^{-1} = \frac{x}{x^2+y^2} = u_x.
\end{aligned}
$$

When $y > 0$, $v(x, y) = \pi/2 - \arctan(x/y)$; when $y < 0$, $v(x, y) = -\pi/2 - \arctan(x/y)$. Every $z \in \Omega$ lies in the interior of one of these 3 domains ($x > 0$, $y > 0$, $y < 0$). If $y > 0$ then

$$
\begin{aligned}
v_x &= -(1/y)(1 + (x/y)^2)^{-1} = \frac{-y}{x^2+y^2} = -u_y, \\
v_y &= (x/y^2)(1 + (x/y)^2)^{-1} = \frac{x}{x^2+y^2} = u_x.
\end{aligned}
$$

If $y < 0$ then v_x and v_y have the same values.

2. For $R > 0$, let γ_R be the path defined by

$$\gamma_R(t) = Re^{it},$$

for $0 \le t \le \pi$. Show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz = 0.$$

(We will use this result later.)

Solution. If $z = Re^{it} = R(\cos t + i \sin t)$ then $e^{iz} = e^{-R\sin t} e^{iR\cos t}$, and $|e^{iz}| = e^{-R\sin t}$. Therefore

$$
\begin{aligned}
\int_{\gamma_R} \frac{e^{iz}}{z}\, dz &= \int_0^\pi \frac{e^{-R\sin t} e^{iR\cos t}}{Re^{it}}\, Rie^{it}\, dt \\
&= i\int_0^\pi e^{-R\sin t} e^{iR\cos t}\, dt,
\end{aligned}
$$

which has absolute value less than or equal to

$$\int_0^\pi e^{-R\sin t}\,dt.$$

We will show that this last quantity goes to 0 in two ways. *Fast way*. For all $0 \le t \le \pi$ and $R > 0$, $e^{-R\sin t} \le 1$. As $R \to \infty$, $e^{-R\sin t} \to 0$ for almost every $t \in [0, \pi]$ (every one except 0 and π). The result now follows by Dominated Convergence. *Slower way*. Let $0 < \epsilon < \pi/2$. Then

$$\int_0^\pi e^{-R\sin t}\,dt \le (I) + (II) + (III),$$

where (I) is the integral from 0 to ϵ, (II) is the integral from ϵ to $\pi - \epsilon$, and (III) is the integral from $\pi - \epsilon$ to π. Since $e^{-R\sin t} \le 1$ on all of $[0, \pi]$, (I) and (III) are both $\le \epsilon$. On the other hand, $e^{-R\sin t} \le e^{-R\sin\epsilon}$ on all of $[\epsilon, \pi - \epsilon]$, implying $(II) \le (\pi - 2\epsilon)e^{-R\sin\epsilon}$, which goes to 0 as $R \to \infty$. Therefore $(I) + (II) + (III) < 3\epsilon$ when R is large enough. QED.

3a) Let S_1 be the square with corners at 0, 1, $1+i$, and i, and let γ_1 be the path that runs around S_1's border once in the counterclockwise direction. Find

$$\int_{\gamma_1} |z|^2\,dz.$$

3b) (This one's harder.) Find

$$\int_{\gamma_1} |z|\,dz.$$

Express your answers to parts a) and b) in exact form, and *do not* express the answer to b) in terms of the ArcSinh (which *Mathematica* will give you).

Solutions. a) The integral is the sum of four integrals, namely

$$\int_0^1 t^2\,dt + i\int_0^1 (1+t^2)\,dt + \int_1^0 (1+t^2)\,dt + i\int_1^0 t^2\,dt.$$

The real parts sum and simplify to

$$\int_0^1 (-1)\,dt = -1,$$

and the imaginary parts sum and simplify to

$$\int_0^1 1\,dt = 1.$$

The integral equals $-1+i$.
b) The integral is the sum of

$$\int_0^1 t\,dt + i\int_0^1 \sqrt{1+t^2}\,dt + \int_1^0 \sqrt{1+t^2}\,dt + i\int_1^0 t\,dt,$$

which simplifies slightly to

$$(1-i)\int_0^1 t\,dt + (i-1)\int_0^1 \sqrt{1+t^2}\,dt = (1-i)\left(\int_0^1 t\,dt - \int_0^1 \sqrt{1+t^2}\,dt\right).$$

The big question is: What is $\int_0^1 \sqrt{1+t^2}\,dt$? Substituting $t=\tan\theta$, $dt=\sec^2\theta$, we get

$$\int \sqrt{1+t^2}\,dt \mapsto \int \sec^3\theta\,d\theta,$$

where I'm using '$\mapsto$' to mean "transforms to". *Mathematica* gives a correct but unintelligible answer to this freshman calculus problem. However, Old Reliable Integration By Parts gives:

$$\begin{aligned}\int \sec^3\theta\,d\theta &= \int \sec\theta\cdot\sec^2\theta\,d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta \\ &= \sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)\,d\theta,\end{aligned}$$

implying

$$\begin{aligned}2\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta + \int \sec\theta \\ &= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C,\end{aligned}$$

and

$$\int \sec^3\theta\,d\theta = (1/2)\,(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) + C.$$

We now reverse the substitution. If $t=\tan\theta$ then $\sqrt{1+t^2}=\sec\theta$, so

$$\int \sqrt{1+t^2}\,dt = (1/2)\left(t\sqrt{1+t^2} + \ln|t+\sqrt{1+t^2}|\right) + C,$$

and

$$\int_0^1 \sqrt{1+t^2}\,dt = (1/2)\left(\sqrt{2} + \ln(1+\sqrt{2})\right).$$

Since $\int_0^1 t\,dt = 1/2$, the integral of $|z|$ around S_1 equals

$$(1-i)\left(1/2-(1/2)\left(\sqrt{2}+\ln(1+\sqrt{2})\right)\right)=\frac{1-i}{2}\left(1-\sqrt{2}-\ln(1+\sqrt{2})\right).$$

4. In class we saw that any convergent power series can be differentiated term by term. Use this fact and the formula for a very familiar power series to show that

$$\sum_1^\infty nz^n = \frac{z}{(1-z)^2}$$

when $|z|<1$. Find analogous expressions for

$$\sum_1^\infty n^2 z^n$$

and

$$\sum_1^\infty n^3 z^n$$

(valid for $|z|<1$), and use them to find the exact values of

$$\sum_1^\infty n^2(1/5)^n$$

and

$$\sum_1^\infty n^3(2/7)^n.$$

Mathematica will find all of these series for you (so you can use it as a check), but I want to see your work.

Solution. The familiar power series is

$$\sum_0^\infty z^n = \frac{1}{1-z},$$

valid for $|z|<1$. Differentiating term by term gives

$$\sum_1^\infty nz^{n-1} = \frac{1}{(1-z)^2}.$$

Multiplying both sides by z yields the first formula. Now differentiate again and multiply by z. You get

$$\sum_1^\infty n^2 z^n = \frac{z+z^2}{(1-z)^3},$$

which *Mathematica* confirms. Repeat the procedure and get

$$\sum_1^\infty n^3 z^n = z\left(\frac{(1+2z)(1-z)+3(z+z^2)}{(1-z)^4}\right) = \frac{z+4z^2+z^3}{(1-z)^4},$$

also confirmed by *Mathematica*. The respective values of the numerical sums are

$$\frac{(1/5)+(1/25)}{(4/5)^3} = \frac{25+5}{64} = \frac{30}{64} = \frac{15}{32}$$

and

$$\frac{(2/7)+4(4/49)+(8/343)}{(5/7)^4} = \frac{686+784+56}{625} = \frac{1526}{625}.$$

5. Let S_2 be the square with corners at $1+i$, $-1+i$, $-1-i$, and $1-i$, and let let γ_2 be the path that runs around S_2's border once in the counterclockwise direction. Show that if $f : \mathbf{C} \mapsto \mathbf{C}$ is any continuous function that depends only on $|z|$ (so, $f(z) = f(|z|)$ for all z), then

$$\int_{\gamma_2} f(z)\, dz = 0.$$

Solution. If we start at the southwest corner, the integral is

$$\int_{-1}^1 f(\sqrt{1+t^2})\, dt + i\int_{-1}^1 f(\sqrt{1+t^2})\, dt + \int_1^{-1} f(\sqrt{1+t^2})\, dt + i\int_1^{-1} f(\sqrt{1+t^2})\, dt.$$

The first and third integrals (i.e., the real ones) cancel each other (they're done in opposite directions), and so do the second and fourth (the imaginary ones, for the same reason).

6. Suppose that $f : \Delta(a;r) \mapsto \mathbf{C}$ is continuous. For $0 < \rho < r$, let γ_ρ be the path defined by $\gamma_\rho(t) = a + \rho e^{2\pi i t}$ for $0 \le t \le 1$. Show that

$$\lim_{\rho\to 0} \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{z-a}\, dz$$

exists and equals $f(a)$.

Solution. After writing everything in terms of γ and γ', the integral in question becomes

$$\frac{1}{2\pi i}\int_0^1 \frac{f(a+\rho e^{2\pi i t})}{\rho e^{2\pi i t}}\, 2\pi i \rho e^{2\pi i t}\, dt = \int_0^1 f(a+\rho e^{2\pi i t})\, dt.$$

Given $\epsilon > 0$, let $0 < \delta < r$ be so small that $|z - a| < \delta$ implies $|f(a) - f(z)| < \epsilon$. If $0 < \rho < \delta$ then

$$\begin{aligned} \left| f(a) - \int_0^1 f(a + \rho e^{2\pi i t})\, dt \right| &= \left| \int_0^1 \left(f(a) - f(a + \rho e^{2\pi i t}) \right) dt \right| \\ &\le \int_0^1 |f(a) - f(a + \rho e^{2\pi i t})|\, dt \\ &< \epsilon, \end{aligned}$$

because $|f(a) - f(a + \rho e^{2\pi i t})| < \epsilon$ for all $t \in [0, 1]$. QED.

7. Let $K \subset \mathbf{C}$ be compact (with respect to the usual, absolute-value metric). Show that $\mathbf{C} \setminus K$ has one—and only one—unbounded component. Give an example of a closed subset E of $\mathbf{C}$ such that $\mathbf{C} \setminus E$ has infinitely many unbounded components.

Solutions. Let $R < \infty$ be so big that $K \subset \overline{\Delta(0; R)}$, and define $U = \mathbf{C} \setminus \overline{\Delta(0; R)}$. The set U is connected (it's path connected). Therefore it is contained in a maximal connected subset (i.e., a component) of $\mathbf{C} \setminus K$. This component must be unbounded, because it contains U. That shows that $\mathbf{C} \setminus K$ has at least *one* unbounded component. Suppose G_1 and G_2 are two unbounded components of $\mathbf{C} \setminus K$. Each G_i contains a z_i such that $|z_i| > R$; hence $G_i \cap U \neq \emptyset$. Therefore $G_i \cup U$ is connected and (since G_i is a maximal connected subset of $\mathbf{C} \setminus K$), $U \subset G_i$, implying $G_1 \cap G_2 \neq \emptyset$ (because the intersection contains U), $G_1 \cup G_2$ is connected, and (by maximality), $G_1 = G_2$. Upshot: $\mathbf{C} \setminus K$ has exactly one unbounded component. For the counterexample, let $E = \cup_{-\infty}^{\infty} \{x + iy :\ x = n\}$. This set is closed because $\mathbf{C} \setminus E = \cup_{-\infty}^{\infty} \{x + iy :\ n - 1 < x < n\}$ is open. The components of $\mathbf{C} \setminus E$ are the vertical strips $\{x + iy :\ n - 1 < x < n\}$, each of which is unbounded. To see they're components, first note that they're connected (they're path connected). Suppose that $\{x + iy :\ n - 1 < x < n\}$ is strictly contained in a component U of $\mathbf{C} \setminus E$. Since $\mathbf{C} \setminus E$ is open, so is U. Therefore U is path connected. Let $z \in U \setminus \{x + iy :\ n - 1 < x < n\}$. Either z's real part is $< n - 1$ or $> n$. By continuity, any path connecting z to a point in $\{x + iy :\ n - 1 < x < n\}$ will have to cross one of the vertical lines $x = n - 1$ or $x = n$—i.e., will leave $\mathbf{C} \setminus E$—which would be a contradiction. Therefore $\{x + iy :\ n - 1 < x < n\}$ is a maximal connected subset of $\mathbf{C} \setminus E$.

8. Give an example of a compact subset K of $\mathbf{C}$ such that $\mathbf{C} \setminus K$ has infinitely many components.

Solution. $K = \{0\} \cup (\cup_1^{\infty} \{z \in \mathbf{C} :\ |z| = 1/n\})$. Best to draw a picture. This is a lot like the set E from the previous problem, with circles (kind of) replacing vertical lines. Notice the importance of including 0 in K.

9. Let γ be the path defined by $\gamma(t) = e^{2\pi i t}$ for $0 \le t \le 1$. For $k = 0, 1, 2, 3, \ldots$, find the value of the integral

$$\int_\gamma \frac{e^{iz}}{z^k}\, dz.$$

Solution. Since e^{iz} is analytic everywhere, the value for $k = 0$ is 0. For $k \neq 0$,

$$\frac{e^{iz}}{z^k} = \sum_{n=-k}^{\infty} \frac{(iz)^{n+k}}{z^k (n+k)!} = \sum_{n=-k}^{\infty} \frac{i^{n+k} z^n}{(n+k)!},$$

which converges uniformly on $\{z : |z| = 1\}$. If $n \neq -1$ then $\int_\gamma z^n \, dz = 0$, while $\int_\gamma z^{-1} \, dz = 2\pi i$, as seen in class. Therefore the integral is

$$\int_\gamma \frac{i^{k-1} z^{-1}}{(k-1)!} \, dz = (2\pi i) i^{k-1} ((k-1)!)^{-1} = \frac{2\pi i^k}{(k-1)!}.$$

10. Show that $f(x+iy) \equiv \sqrt{|xy|}$ satisfies the Cauchy-Riemann equations at 0 but is NOT analytic there.

Solution. We have $u(x,y) = \sqrt{|xy|}$ and $v(x,y) \equiv 0$. Since $u(x,0) \equiv u(0,y) \equiv 0$, $u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$. But

$$\lim_{t \to 0^+} \frac{f(t+it) - f(0)}{t+it} = \lim_{t \to 0^+} \frac{t}{t+it} = \frac{1}{1+i},$$

while

$$\lim_{t \to 0^-} \frac{f(t+it) - f(0)}{t+it} = \lim_{t \to 0^-} \frac{-t}{t+it} = \frac{-1}{1+i},$$

showing that $f'(0)$ does not exist.