Math 331 Homework #4 **Spring, 2010**

Assigned March 22. Due March 33 (i.e., April 2). The seven numbered problems are worth 14 points each, with 2 points added to make 100.

1. For $a \in \mathbf{C}$ and $r > 0$, let $f : \Delta(a; r) \mapsto \mathbf{C}$ be analytic. Show that, if $m \geq 1$ is an integer, then f has a zero of order m at a if and only if $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$.

2a) Let $\Omega \subset \mathbf{C}$ be a region and suppose that $f : \Omega \mapsto \mathbf{C}$ is analytic. Show that if $\bar{f}$ is analytic on any non-empty open subset of Ω then f is constant.
2b) Let $\Omega \subset \mathbf{C}$ be a region, and suppose that f and g are both analytic on Ω. Show that if $\bar{f}g$ is analytic on Ω then either f is constant or g is identically 0.

3. Let $\gamma(t) = 1 + e^{2\pi i t}$ for $0 \leq t \leq 1$. Find a formula for

$$\int_\gamma \left(\frac{z}{z-1} \right)^n dz,$$

valid for all positive integers n.

4. Show that if $f : \mathbf{C} \mapsto \mathbf{C}$ is continuous, and f is analytic on $\mathbf{C} \setminus [0, 1]$, then f is analytic on all of $\mathbf{C}$.

5. Suppose that $\psi : [0, 1] \mapsto \mathbf{R}$ is a smooth function such that $\int_0^1 \psi'(s)\, ds = 0$. For $A \in \mathbf{R}$, define $\gamma_A(s) \equiv e^{iA\psi(s)}$. Show that, for all A, γ_A is a smooth closed curve in $\Omega \equiv \mathbf{C} \setminus \{0\}$ and $\gamma_A \sim 0$.

6. Let $\Omega \subset \mathbf{C}$ be a region, and suppose that f and g are two functions that are analytic on Ω. Show that if $fg \equiv 0$ (i.e., $f(z)g(z) = 0$ for all $z \in \Omega$), then either $f \equiv 0$ or $g \equiv 0$.

7a) Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that

$$|f(z)| \leq \frac{88 + 112\sqrt{|z|}}{17 + 29|z|}$$

for all z. Show that f is constant, and find the constant.
7b) Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that $|f(z)| \leq 12{,}358(13 + |z|)^{2\pi}$ for all z. Show that f is a polynomial of degree at most 6.