Math 331 Homework #4 Solutions **Spring, 2010**

Assigned March 22. Due March 33 (i.e., April 2). The seven numbered problems are worth 14 points each, with 2 points added to make 100.

1. For $a \in \mathbf{C}$ and $r > 0$, let $f : \Delta(a; r) \mapsto \mathbf{C}$ be analytic. Show that, if $m \geq 1$ is an integer, then f has a zero of order m at a if and only if $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$.

Solution. f has a zero of order $m \geq 1$ at a if and only if $f(z) = (z-a)^m g(z)$, where g is analytic on $\Delta(a; r)$ and $g(a) \neq 0$. We can expand g in a power series, convergent in $\Delta(a; r)$,

$$g(z) = \sum_0^\infty b_n (z-a)^n, \tag{1}$$

and such that $b_0 \neq 0$. Then f has a zero of order $m \geq 1$ if and only if

$$f(z) = (z-a)^m \sum_0^\infty b_n (z-a)^n \tag{2}$$

where the series is as in (1). However, we can rewrite the right-hand side of (2) as

$$\sum_0^\infty b_n (z-a)^{m+n} = \sum_0^\infty a_n (z-a)^n,$$

where

$$a_n = \begin{cases} 0 & \text{if } n < m; \\ b_{n-m} & \text{if } n \geq m. \end{cases}$$

But then $a_n = f^{(n)}(a)/n!$, which equals 0 for $n < m$ and is non-zero for $n = m$. Conversely, if $f^{(n)}(a) = 0$ for all $n < m$ and $f^{(m)}(a) \neq 0$, we can reverse these implications and get $f = (z-a)^m g$, where g has the desired form. QED.

2a) Let $\Omega \subset \mathbf{C}$ be a region and suppose that $f : \Omega \mapsto \mathbf{C}$ is analytic. Show that if $\bar{f}$ is analytic on any non-empty open subset of Ω then f is constant.
2b) Let $\Omega \subset \mathbf{C}$ be a region, and suppose that f and g are both analytic on Ω. Show that if $\bar{f}g$ is analytic on Ω then either f is constant or g is identically 0.

Solutions. a) Suppose that f and $\bar{f}$ are both analytic on $\Delta(a; r) \subset \Omega$. Write $f = u + iv$, $\bar{f} = u - iv$. Both must satisfy the Cauchy-Riemann equations on $\Delta(a; r)$. A little algebra yields that $u_x = u_y = v_x = v_y \equiv 0$ there, implying $f' \equiv 0$ on the disk, and therefore (since Ω is a region), $f' \equiv 0$ on Ω, and f is constant. b) If $g \not\equiv 0$ there is a disk $\Delta(a; r) \subset \Omega$ on which g is never 0. Therefore

$$\frac{\bar{f}g}{g} = \bar{f}$$

is analytic there, implying that f is constant.

3. Let $\gamma(t) = 1 + e^{2\pi i t}$ for $0 \leq t \leq 1$. Find a formula for

$$\int_\gamma \left(\frac{z}{z-1} \right)^n dz,$$

valid for all positive integers n.

Solution. There are several ways to do this. Here's one. Set $f(z) = z^n$. Then

$$\frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{(z-1)^n} \, dz = f^{(n-1)}(1) = n!,$$

implying

$$\int_\gamma \frac{f(z)}{(z-1)^n} \, dz = 2\pi i n.$$

4. Show that if $f : \mathbf{C} \mapsto \mathbf{C}$ is continuous, and f is analytic on $\mathbf{C} \setminus [0,1]$, then f is analytic on all of $\mathbf{C}$.

Solution. By Morera's Theorem, it suffices to show $\int_T f \, dz = 0$ for all triangles T. The details of this were really shown in the class on 3/25. If T doesn't touch or enclose $[0,1]$, no problem (as done in class). If T touches $[0,1]$ at a corner but doesn't enclose or cross it, then we get $\int_T f \, dz = 0$, as in class. If a side of T lies entirely on $[0,1]$, we proceed as follows. WLOG, assume that T has corners ζ, x_1, and x_2, where $\Im(\zeta) > 0$ and $0 \leq x_1 < x_2 \leq 1$. For every $0 < \epsilon < \Im(\zeta)$, let $\gamma_{1,\epsilon}$ be the triangular path that runs along the straight line from ζ to x_1, but makes a left turn when it crosses $\Im(z) = \epsilon$ (at a point we'll call a_ϵ), to go horizontally until it reaches the line segment connecting ζ and x_2 (at a point we'll call b_ϵ), and then runs along that line segment to close up the triangle at ζ. Let $\gamma_{2,\epsilon}$ be the parallelogram path that starts at x_1, runs straight to x_2, follows a straight line to b_ϵ, follows a second straight line to a_ϵ, and then goes straight back to x_1. We notice that

$$\int_T f(z) \, dz = \int_{\gamma_{1,\epsilon}} f(z) \, dz + \int_{\gamma_{1,\epsilon}} f(z) \, dz$$

for all ϵ. The integral

$$\int_{\gamma_{1,\epsilon}} f(z) \, dz$$

equals 0 for all ϵ. The integral

$$\int_{\gamma_{2,\epsilon}} f(z) \, dz$$

can be expressed as

$$\int_{x_1 + c_\epsilon}^{x_2 - d_\epsilon} (f(t) - f(t + i\epsilon)) \, dt \qquad (3)$$

(where c_ϵ and d_ϵ are small non-negative numbers depending on ϵ), PLUS the integrals over at most FOUR line segments, the sum of whose lengths is $\leq$ a fixed constant times ϵ. The continuity of f (implying its boundedness near $[0,1]$) now implies that the integral over the parallelogram goes to 0 as $\epsilon \to 0$. (The continuity of f makes (3) $\to 0$.) Therefore $\int_T f\,dz = 0$. The remaining cases, where T encloses or crosses $[0,1]$, or has only part of a side on $[0,1]$, are reducible to these by decomposing T, more or less as done in class.

5. Suppose that $\psi : [0,1] \mapsto \mathbf{R}$ is a smooth function such that $\int_0^1 \psi'(s)\,ds = 0$. For $A \in \mathbf{R}$, define $\gamma_A(s) \equiv e^{iA\psi(s)}$. Show that, for all A, γ_A is a smooth closed curve in $\Omega \equiv \mathbf{C} \setminus \{0\}$ and $\gamma_A \sim 0$.

Solution. The hypothesis on ψ implies $\psi(0) = \psi(1)$, therefore $\gamma_A(0) = \gamma_A(1)$. The smoothness of γ_A follows from ψ's smoothness. $\gamma_A \sim 0$ via this homotopy:

$$\Gamma(s,t) \equiv e^{i(1-t)A\psi(s)}.$$

Γ is clearly continuous and maps into Ω. $\Gamma(s,0) \equiv \gamma_A(s)$, $\Gamma(s,1) \equiv 1$, and, since $\Gamma(s,t) \equiv \gamma_{(1-t)A}(s)$, all the $\Gamma(\cdot,t)$'s are closed curves.

6. Let $\Omega \subset \mathbf{C}$ be a region, and suppose that f and g are two functions that are analytic on Ω. Show that if $fg \equiv 0$ (i.e., $f(z)g(z) = 0$ for all $z \in \Omega$), then either $f \equiv 0$ or $g \equiv 0$.

Solution. Suppose that $f \not\equiv 0$. Then there is a disk $\Delta(a;r) \subset \Omega$ on which f is never 0, forcing g to be 0 on *all* of $\Delta(a;r)$, which (because Ω is a region), implies that $g \equiv 0$.

7a) Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that

$$|f(z)| \leq \frac{88 + 112\sqrt{|z|}}{17 + 29|z|}$$

for all z. Show that f is constant, and find the constant.

7b) Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that $|f(z)| \leq 12{,}358(13 + |z|)^{2\pi}$ for all z. Show that f is a polynomial of degree at most 6.

Solutions. a) We observe that

$$\lim_{|z|\to\infty} f(z) = 0.$$

Let R be so large that $|f(z)| \leq 1$ for all z with $|z| > R$, and let $M = \sup\{|f(z)| : \ |z| \leq R\}$, which is finite because f is continuous and $\{z : \ |z| \leq R\}$ is compact. Then $|f(z)| \leq \max(M,1)$ for all z, implying f is constant. Since f goes to 0 at infinity, the constant has to be 0. b) Write f as a power series centered at 0:

$$f(z) = \sum_0^\infty a_n z^n.$$

Let

$$M_R = \sup\{|f(z)| : \ |z| = R\}.$$

We have that

$$M_R \leq 12,358(13+R)^{2\pi}.$$

By the Cauchy estimates,

$$|a_n| \leq \frac{M_R}{R^n}$$

for all n any $R > 0$. But $M_R/R^n \to 0$ as $R \to \infty$ if $n \geq 7$, because $2\pi < 7$. Therefore $a_n = 0$ for all $n > 6$, proving the result.