Math 331 Homework #5 **Spring, 2010**

Assigned April 7. Due April 16. The seven numbered problems are worth 14 points each, with 2 points added to make 100.

1. Show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\, dx = \frac{\pi(a+1)e^{-a}}{2}$$

whenever $a \geq 0$.

2. Show that, if n is any integer strictly bigger than 0, then

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

3. Show that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx = \pi/\sqrt{2}.$$

4. Define $f(z) = \log(1+z)$ for $|z| < 1$, where we choose the branch that makes $f(0) = 0$. Find f's power series expansion around $z = 0$ and determine its radius of convergence.

5. Show that if $\{c_n\} \subset \mathbf{C}$ is any sequence of complex numbers such that $c_n \to c \in \mathbf{C}$, then

$$\lim_{n\to\infty} \left(1 + \frac{c_n}{n}\right)^n = e^c.$$

Hint: Problem #4 might help here!

6. Let $f(z) = (z^2 - z - 6)^{-1}$ on $\mathbf{C} \setminus \{-2, 3\}$. Find f's power series expansion around 0, valid for in the disk $\Delta(0; 2)$, and also find its Laurent expansions, valid in the annuli $Ann(0; 2, 3)$ and $Ann(0; 3, \infty)$. You might save time by first finding formulas for the power series or Laurent expansions of $(z-a)^{-1}$ ($a \neq 0$), valid in $\Delta(0; |a|)$ or $Ann(0; |a|, \infty)$.

7. Let $D \equiv \{z \in \mathbf{C} : \ |z| < 1\}$, and define $\mathcal{B}$ to be the family of analytic functions f that map from D into D. Suppose that $\phi : D \mapsto D$ has the property that, for every *three* points z_1, z_2, and z_3 in D, there is an $f \in \mathcal{B}$ (depending on the points!) such that $\phi(z_1) = f(z_1)$, $\phi(z_2) = f(z_2)$, and $\phi(z_3) = f(z_3)$. Show that $\phi \in \mathcal{B}$; i.e., that $\phi'(z)$ exists for all $z \in D$. Hint: Suppose it's false for $z = 0$. Then, for some $\epsilon > 0$, there are sequences $\{z_n\}_n$ and $\{w_n\}_n$, converging to 0, such that

$$\left| \frac{\phi(z_n) - \phi(0)}{z_n} - \frac{\phi(w_n) - \phi(0)}{w_n} \right| > \epsilon$$

for all n. The hypothesis on ϕ implies that, for each n, there is a function $f_n \in \mathcal{B}$ such that

$$\left| \frac{f_n(z_n) - f_n(0)}{z_n} - \frac{f_n(w_n) - f_n(0)}{w_n} \right| > \epsilon. \tag{1}$$

Now show that (1) leads to a contradiction. (The Cauchy Integral Formula can help here.) Then show how to adapt the argument to a generic $z \in D$.