Math 331 Homework #5 Solutions **Spring, 2010**

Assigned April 7. Due April 16. The seven numbered problems are worth 14 points each, with 2 points added to make 100.

1. Show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx = \frac{\pi(a+1)e^{-a}}{2}$$

whenever $a \geq 0$.

Solution. As with the example worked in class, we will integrate

$$f(z) \equiv \frac{e^{iaz}}{(1+z^2)^2}$$

along the contour γ_R (with $R > 1$), which runs along the real axis from $-R$ to R, and then counterclockwise along a semicircular arc from R back to $-R$. For any $R > 1$,

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f; i);$$

and, as with the example from class,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz,$$

implying

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(1+x^2)^2}\,dx = 2\pi i \operatorname{Res}(f; i).$$

Therefore everything comes down to figuring out the residue. The function f has a double pole at i, where it can be written

$$f(z) = (z-i)^{-2} g(z)$$

with $g(z) = e^{iaz}/(z+i)^2$. The residue equals $g'(i)$. By the quotient rule,

$$g'(z) = \frac{iae^{iaz}(z+i)^2 - 2e^{iaz}(z+i)}{(z+i)^4} = e^{iaz}\left(\frac{ia(z+i)-2}{(z+i)^3}\right);$$

therefore

$$g'(i) = e^{-a}\left(\frac{-2a-2}{-8i}\right) = e^{-a}\left(\frac{a+1}{4i}\right),$$

and the integral is

$$2\pi i e^{-a}\left(\frac{a+1}{4i}\right) = \frac{\pi(a+1)e^{-a}}{2},$$

as claimed.

2. Show that, if n is any integer strictly bigger than 0, then

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \pi \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

Solution. The integral equals $2\pi i$ times the residue of $f(z)$ at $z = i$, where $f(z) = 1/(z^2+1)^{n+1}$. Since f has a pole of order $n+1$ there, its residue is

$$\frac{g^{(n)}(i)}{n!}, \qquad (\alpha)$$

where $g(z) = (z+i)^{-n-1}$. We need to find (α). Now,

$$\begin{aligned} g^{(n)}(z) &= (-1)^n(n+1)(n+2)\cdots(n+n)(z+i)^{-2n-1} \\ &= (-1)^n \frac{(2n)!}{n!(z+i)^{2n+1}}. \qquad (\beta) \end{aligned}$$

Therefore

$$\frac{g^{(n)}(i)}{n!} = (-1)^n \frac{(2n)!}{(n!)^2(2i)^{2n+1}}.$$

Now, $(2i)^{2n+1} = (-1)^n 2^{2n}(2i)$, $(2n)! = 2^n(n!)(1\cdot 3\cdot 5\cdots(2n-1))$, and $2^n(n!) = 2\cdot 4\cdot 6\cdots(2n)$. Therefore ($\beta$) simplifies to

$$\frac{(-1)^n 2^n(n!)(1\cdot 3\cdot 5\cdots(2n-1))}{(2i)(-1)^n 2^n(n!)(2\cdot 4\cdot 6\cdots(2n))} = \frac{(1\cdot 3\cdot 5\cdots(2n-1))}{(2i)(2\cdot 4\cdot 6\cdots(2n))},$$

which, when multiplied by $2\pi i$, gives the desired value of the integral.

3. Show that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx = \pi/\sqrt{2}.$$

Solution. Put $f(z) = z^2/(z^4+1)$. This function has poles of order 1 at $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, and $e^{7\pi i/4}$. The integral we seek will equal $2\pi i$ times the sum of f's residues at $e^{\pi i/4}$ and $e^{3\pi i/4}$. The denominator of f factors into:

$$\begin{aligned} z^4+1 &= [(z^2-\sqrt{2}z+1)][(z^2+\sqrt{2}z+1)] \\ &= [(z-e^{\pi i/4})(z-e^{7\pi i/4})][(z-e^{3\pi i/4})(z-e^{5\pi i/4})], \end{aligned}$$

which we get from the quadratic formula. At $z = e^{\pi i/4}$, f has residue equal to

$$\begin{aligned} \frac{(e^{\pi i/4})^2}{[(e^{\pi i/4}-e^{7\pi i/4})][(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{5\pi i/4})]} &= \frac{i}{[(i\sqrt{2})(\sqrt{2})(1+i)(\sqrt{2})]} \\ &= \frac{1-i}{4\sqrt{2}}. \end{aligned}$$

At $z = e^{3\pi i/4}$, f has residue equal to

$$\frac{(e^{3\pi i/4})^2}{[(e^{3\pi i/4} - e^{5\pi i/4})][(e^{3\pi i/4} - e^{\pi i/4})(e^{3\pi i/4} - e^{7\pi i/4})]} = \frac{-i}{[(i\sqrt{2})(-\sqrt{2})(\sqrt{2})(-1+i)]} = \frac{-1-i}{4\sqrt{2}}.$$

The sum of the residues is

$$\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}}.$$

The value of the integral is $2\pi i$ times this, or $\pi/\sqrt{2}$.

4. Define $f(z) = \log(1+z)$ for $|z| < 1$, where we choose the branch that makes $f(0) = 0$. Find f's power series expansion around $z = 0$ and determine its radius of convergence.

Solution. $f'(z) = 1/(1+z) = \sum_0^\infty (-1)^n z^n$, therefore

$$f(z) = f(0) + \sum_0^\infty (-1)^n z^{n+1}/(n+1) = \sum_1^\infty (-1)^{n+1} \frac{z^n}{n} = z - z^2/2 + z^3/3 - z^4/4 + \cdots,$$

which has radius of convergence 1, since $n^{1/n} \to 1$ as $n \to \infty$.

5. Show that if $\{c_n\} \subset \mathbf{C}$ is any sequence of complex numbers such that $c_n \to c \in \mathbf{C}$, then

$$\lim_{n\to\infty} \left(1 + \frac{c_n}{n}\right)^n = e^c.$$

Hint: Problem #4 might help here!

Solution. WLOG, $|c_n/n| < 1$ for all n. Therefore

$$\left(1 + \frac{c_n}{n}\right) = \exp(\log(1 + c_n/n)),$$

where the logarithm is the same as in problem #4. But

$$\log(1 + c_n/n) = c_n/n + O((1/n)^2),$$

by problem #4. Therefore

$$\left(1 + \frac{c_n}{n}\right)^n = \exp(n \log(1 + c_n/n)) = \exp(c_n) \exp(n\, O((1/n)^2)) = \exp(c_n) \exp(O(1/n)) \to e^c e^0 = e^c.$$

6. Let $f(z) = (z^2 - z - 6)^{-1}$ on $\mathbf{C} \setminus \{-2, 3\}$. Find f's power series expansion around 0, valid for in the disk $\Delta(0;2)$, and also find its Laurent expansions, valid in the annuli $Ann(0;2,3)$ and $Ann(0;3,\infty)$. You might save time by first finding formulas for the power series or Laurent expansions of $(z-a)^{-1}$ $(a \neq 0)$, valid in $\Delta(0;|a|)$ or $Ann(0;|a|,\infty)$.

Solutions. Follow the hint. If $|z| < |a|$ then

$$\begin{aligned} \frac{1}{z-a} &= \frac{-1}{a}\frac{1}{1-z/a} \\ &= \frac{-1}{a}\sum_0^\infty (z/a)^n \\ &= -\sum_0^\infty z^n a^{-n-1}. \end{aligned}$$

If $|z| > |a|$ then

$$\begin{aligned} \frac{1}{z-a} &= \frac{1}{z}\frac{1}{1-a/z} \\ &= \frac{1}{z}\sum_0^\infty (a/z)^n \\ &= \sum_0^\infty a^n z^{-n-1} \\ &= \sum_{-\infty}^{-1} z^n a^{-n-1}. \end{aligned}$$

Now,

$$\frac{1}{z^2-z-6} = (1/5)\left(\frac{1}{z-3} - \frac{1}{z+2}\right).$$

Therefore, if $|z| < 2$,

$$f(z) = (1/5)\sum_0^\infty z^n \left((-2)^{-n-1} - 3^{-n-1}\right);$$

if $2 < |z| < 3$,

$$f(z) = -(1/5)\sum_{-\infty}^{-1} z^n(-2)^{-n-1} - (1/5)\sum_0^\infty z^n 3^{-n-1};$$

if $|z| > 3$,

$$f(z) = -(1/5)\sum_{-\infty}^{-1} z^n \left((-2)^{-n-1} - 3^{-n-1}\right).$$

7. Let $D \equiv \{z \in \mathbf{C} : |z| < 1\}$, and define $\mathcal{B}$ to be the family of analytic functions f that map from D into D. Suppose that $\phi : D \mapsto D$ has the property that, for every *three* points z_1, z_2, and z_3 in D, there is an $f \in B$ (depending on the points!) such that $\phi(z_1) = f(z_1)$, $\phi(z_2) = f(z_2)$, and $\phi(z_3) = f(z_3)$. Show that $\phi \in \mathcal{B}$; i.e., that $\phi'(z)$ exists for all $z \in D$. Hint: Suppose it's false for $z = 0$. Then, for some $\epsilon > 0$, there are sequences $\{z_n\}_n$ and $\{w_n\}_n$, converging to 0, such that

$$\left| \frac{\phi(z_n) - \phi(0)}{z_n} - \frac{\phi(w_n) - \phi(0)}{w_n} \right| > \epsilon$$

for all n. The hypothesis on ϕ implies that, for each n, there is a function $f_n \in \mathcal{B}$ such that

$$\left| \frac{f_n(z_n) - f_n(0)}{z_n} - \frac{f_n(w_n) - f_n(0)}{w_n} \right| > \epsilon \tag{1}$$

Now show that (1) leads to a contradiction. (The Cauchy Integral Formula can help here.) Then show how to adapt the argument to a generic $z \in D$.

Solution. We follow the hint. There exist sequences $\{z_n\}$ and $\{w_n\}$, both converging to 0, and there exists a sequence $\{f_n\}_n \subset \mathcal{B}$, such that (1) holds for all n. WLOG, $|z_n|$ and $|w_n|$ are all $< 1/2$. Define $\gamma(s) = (1/2)e^{2\pi i s}$ for $0 \leq s \leq 1$. For all n,

$$\frac{f_n(z_n) - f_n(0)}{z_n} = \frac{1}{2\pi i z_n} \int_\gamma f_n(\zeta) \left(\frac{1}{\zeta - z_n} - \frac{1}{\zeta} \right) d\zeta, \tag{2}$$

and

$$\frac{f_n(w_n) - f_n(0)}{w_n} = \frac{1}{2\pi i w_n} \int_\gamma f_n(\zeta) \left(\frac{1}{\zeta - w_n} - \frac{1}{\zeta} \right) d\zeta. \tag{3}$$

Algebra shows that the right-hand side of (2) is

$$\frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z_n)\zeta} \, d\zeta$$

and the right-hand side of (3) is

$$\frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - w_n)\zeta} \, d\zeta.$$

The difference of the two equals

$$\frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)(w_n - z_n)}{\zeta(\zeta - w_n)(\zeta - z_n)} \, d\zeta. \tag{4}$$

However, because $|f_n| \leq 1$, the integral (4) has absolute value no bigger than

$$\frac{|z_n - w_n|}{((1/2) - |w_n|)((1/2) - |z_n|)},$$

which goes to 0 as $n \to \infty$. Therefore (1) is impossible. For the general case, if $\phi'(z)$ does not exist, there is an $\epsilon > 0$ and there exist sequences $z_n \to z$ and $w_n \to z$ so that

$$\left| \frac{\phi(z_n) - \phi(z)}{z_n - z} - \frac{\phi(w_n) - \phi(z)}{w_n - z} \right| > \epsilon$$

for all n. Thus, for each n, there is a function $f_n \in \mathcal{B}$ such that

$$\left| \frac{f_n(z_n) - f_n(z)}{z_n - z} - \frac{f_n(w_n) - f_n(z)}{w_n - z} \right| > \epsilon. \tag{5}$$

Pick $\rho > 0$ such that $\Delta(z; 2\rho) \subset D$, and define $\tilde{\gamma}(s) = z + \rho e^{2\pi i s}$ for $0 \leq s \leq 1$. WLOG, $|z_n - z|$ and $|w_n - z|$ are all $< \rho$. By the Cauchy Integral Formula,

$$\frac{f_n(z_n) - f_n(z)}{z_n - z} = \frac{1}{2\pi i(z_n - z)} \int_{\tilde{\gamma}} f_n(\zeta) \left(\frac{1}{\zeta - z_n} - \frac{1}{\zeta - z} \right) d\zeta,$$

and

$$\frac{f_n(w_n) - f_n(z)}{w_n - z} = \frac{1}{2\pi i(w_n - z)} \int_{\tilde{\gamma}} f_n(\zeta) \left(\frac{1}{\zeta - w_n} - \frac{1}{\zeta - z} \right) d\zeta,$$

which equal, respectively,

$$\frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f_n(\zeta)}{(\zeta - z_n)(\zeta - z)} \, d\zeta$$

and

$$\frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f_n(\zeta)}{(\zeta - w_n)(\zeta - z)} \, d\zeta.$$

Their difference is

$$\frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f_n(\zeta)(w_n - z_n)}{(\zeta - z)(\zeta - w_n)(\zeta - z_n)} \, d\zeta. \tag{6}$$

However, because $|f_n| \leq 1$, the integral (6) has absolute value no bigger than

$$\frac{|z_n - w_n|}{(\rho - |w_n - z|)(\rho - |z_n - z|)},$$

which goes to 0 as $n \to \infty$, contradicting (5) and proving the result.