Math 331 Homework #6 **Spring, 2010**

Assigned April 19. Due May 3. The ten problems are worth 20 points each, for a total of 200. In all of the problems, D denotes the unit disk $\Delta(0;1)$ and $\mathcal{H}$ is the right half-plane $\{z \in \mathbf{C} : \Re(z) > 0\}$.

1. Let $G_1 = \mathbf{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find an $f \in H(G_1)$ that maps G_1 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

2. Let $G_2 = D \cap \mathcal{H}$. Find an $f \in H(G_2)$ that maps G_2 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

3. Let G_3 be $\mathbf{C} \setminus \mathcal{S}$, where $\mathcal{S}$ (considered as a set in $\mathbf{R}^2$) is the union of $\{0\}$ and the infinite spiral with polar equation $r = e^\theta$. (The inclusion of $\{0\}$ makes $\mathcal{S}$ a closed set.) Find an $f \in H(G_3)$ that maps G_3 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps. Hint: Where does the exponential function send the line with equation $y = mx$?

4. Let $G_4 = \{z = x + iy : \ y < x^2\}$. Find an $f \in H(G_4)$ that maps G_4 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

5. Let $G_5 = \Delta(i; 2) \cap \Delta(-i; 2)$. Find an $f \in H(G_5)$ that maps G_5 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

6a) Suppose f is an entire function such that $|\Im(f(z))| < \exp(|\Re(f(z))|)$ for all z. Show that f is constant.
6b) Suppose f is an entire function such that $|\Im(f(z))\Re(f(z))| < 1$ for all z. Show that f is constant.

7. Suppose that f is an entire function whose range is disjoint from $[0, 1]$ (i.e., $f[\mathbf{C}] \cap [0, 1] = \emptyset$). Show that f is constant.

8. Suppose that $f : \overline{D} \mapsto \mathbf{C}$ is continuous, f is analytic on D, and $|f| \equiv 1$ on ∂D. Show that either f is constant or f has a zero in D.

9. Let Q be the open square with corners at $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$; i.e., $Q = \{z \in \mathbf{C} : |\Re(z)| < 1\} \cap \{z \in \mathbf{C} : |\Im(z)| < 1\}$. Suppose that $f \in H(Q)$ maps Q one-to-one and onto D, and also satisfies $f(0) = 0$. Show that, for all $z \in Q$, $f(iz) = if(z)$.

10. Suppose that $f : D \mapsto D$ is analytic and has zeroes at the points a_1, a_2, ... , a_n in D. Show that, for all $z \in D$,

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - a_k}{1 - z\bar{a_k}} \right|.$$

Hint: If $\alpha \in D$, what is the absolute value of

$$\frac{z - \alpha}{1 - z\bar{\alpha}}$$

when $|z| = 1$?