Math 331 Homework #6 Solutions **Spring, 2010**

Assigned April 19. Due May 3. The ten problems are worth 20 points each, for a total of 200. In all of the problems, D denotes the unit disk $\Delta(0;1)$ and $\mathcal{H}$ is the right half-plane $\{z \in \mathbf{C} : \Re(z) > 0\}$.

1. Let $G_1 = \mathbf{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Find an $f \in H(G_1)$ that maps G_1 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

Solution. The chain $z \mapsto z + 1 \mapsto \sqrt{z+1} \mapsto \sqrt{2}/\sqrt{z+1}$ takes G_1 one-to-one and onto $\mathcal{H} \setminus (0, 1]$. In class we saw how to map $\mathcal{H} \setminus (0, 1]$ one-to-one and onto $\mathcal{H}$. By the Kevin Bacon Principle, we're done.

2. Let $G_2 = D \cap \mathcal{H}$. Find an $f \in H(G_2)$ that maps G_2 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

Solution. The map z^2 takes G_2 one-to-one and onto $D\setminus(-1, 0]$. The function $(1+z)/(1-z)$ takes $D \setminus (-1, 0]$ one-to-one and onto $\mathcal{H} \setminus (0, 1]$. KBP again!

3. Let G_3 be $\mathbf{C}\setminus\mathcal{S}$, where $\mathcal{S}$ (considered as a set in $\mathbf{R}^2$) is the union of $\{0\}$ and the infinite spiral with polar equation $r = e^{\theta}$. (The inclusion of $\{0\}$ makes $\mathcal{S}$ a closed set.) Find an $f \in H(G_3)$ that maps G_3 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps. Hint: Where does the exponential function send the line with equation $y = mx$?

Solution. Follow the hint. The exponential map takes the line $t + imt$ $(-\infty < t < \infty)$ one-to-one and onto the spiral $r = e^{\theta/m}$. But it takes the line $t + imt + 2\pi i$ onto the same spiral, and it's one-to-one on the space (a slanted strip) between them (because no two distinct points in the strip have a difference of $2k\pi i$). Pick $m = 1$ and call this slanted strip I. A branch of the logarithm (the inverse of the exponential) takes G_3 one-to-one and onto I. The mapping $z \mapsto \sqrt{2}e^{-i\pi/4}z - \pi i$ takes I one-to-one and onto the horizontal strip $\{z = x + iy : -\pi < y < \pi\}$, which we looked at in class. KBP!

4. Let $G_4 = \{z = x + iy : y < x^2\}$. Find an $f \in H(G_4)$ that maps G_4 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

Solution. This one's trickier; we'll go at it backwards. Let $A > 0$, to be determined shortly. The map $z \mapsto z + A$ takes $\mathcal{H}$ to $\{z = x + iy : x > A\}$, and then $z \mapsto z^2$ takes $\{z = x + iy : x > A\}$ one-to-one and onto $\{z = x + iy : x > A^2 - y^2/(4A^2)\}$ (look where the mapping sends the line $A+it$). The map $z \mapsto -iz$ (rotation by -90 degrees) takes that domain to $\{z = x + iy : y < -A^2 + x^2/(4A^2)\}$. The mapping $z \mapsto z + iA^2$ takes us to $\{z = x + iy : y < x^2/(4A^2)\}$. Choosing $A = 1/2$ makes this set G_4. To go from G_4 to $\mathcal{H}$, we reverse the steps: $z \mapsto z - i/4 \mapsto i(z - i/4) \mapsto \sqrt{i(z - i/4)} \mapsto -(1/2) + \sqrt{i(z - i/4)}$.

5. Let $G_5 = \Delta(i; 2) \cap \Delta(-i; 2)$. Find an $f \in H(G_5)$ that maps G_5 one-to-one and onto $\mathcal{H}$. Express f as a composition of simpler maps.

Solution. The region G_5 is lens-shaped, with corners at $\pm\sqrt{3}$. At $z = \sqrt{3}$, the upper part of ∂G_5 is perpendicular to the line segment $[-i, \sqrt{3}]$. However, by trigonometry, that line segment makes an angle of 30 degrees with the real axis. *Therefore* the upper part of ∂G_5

meets the real axis at $\sqrt{3}$ with an angle of 60 degrees; and therefore, at $\pm\sqrt{3}$, the arcs of ∂G_5 meet at angles of 120 degrees $= 2\pi/3$ radians. The mapping $(\sqrt{3}+z)^{-1}$ sends $-\sqrt{3}$ to ∞ and sends $\sqrt{3}$ to $(2\sqrt{3})^{-1}$. The line segment $(-\sqrt{3},\sqrt{3})$ goes to $((2\sqrt{3})^{-1},\infty)$. By symmetry and conformality (angle-preserving-ness), ∂G_5 goes to 2 rays emanating from $(2\sqrt{3})^{-1}$. One ray is inclined upward (positive slope) at an angle of 60 degrees ($=\pi/3$ radians) and the other is inclined downward (negative slope) at an angle of 60 degrees. If we apply the map $z-(2\sqrt{3})^{-1}$, we get a symmetric sector of size $2\pi/3$ radians, with vertex at the origin. Now $z \mapsto z^{3/2}$ takes this sector to $\mathcal{H}$.

6a) Suppose f is an entire function such that $|\Im(f(z))| < \exp(|\Re(f(z))|)$ for all z. Show that f is constant.
6b) Suppose f is an entire function such that $|\Im(f(z))\Re(f(z))| < 1$ for all z. Show that f is constant.

Solutions. 6a) Let $\delta =$ the distance from $(0,2)$ to the graph of $y = \exp(|x|)$. It's positive. Therefore

$$h(z) \equiv \frac{1}{f(z)-2i}$$

is a bounded entire function, and hence constant. 6b) Let $\delta =$ the distance from $(2,2)$ to the graph of $|xy| = 1$. It's positive. Therefore

$$h(z) \equiv \frac{1}{f(z)-(2+2i)}$$

is a bounded entire function, and hence constant.

7. Suppose that f is an entire function whose range is disjoint from $[0,1]$ (i.e., $f[\mathbf{C}]\cap[0,1] = \emptyset$). Show that f is constant.

Solution. Since f is never 0, $1/f$ is also entire. But then $1/f$ lies in $\mathbf{C}\setminus[1,\infty)$, which can be mapped one-to-one and onto D by some analytic ϕ. The function $\phi(1/f)$ is bounded and entire, hence constant, forcing f to be constant.

8. Suppose that $f : \overline{D} \mapsto \mathbf{C}$ is continuous, f is analytic on D, and $|f| \equiv 1$ on ∂D. Show that either f is constant or f has a zero in D.

Solution. If f has no zero in D then $1/f$ is also analytic on D and continuous on $\overline{D}$. By the Maximum Principle, $|f| \le 1$ in all of $\overline{D}$. Likewise, $|1/f| \le 1$ in $\overline{D}$. Putting them together implies $|f| \equiv 1$ in $\overline{D}$ which (by the Maximum Principle again) implies f is constant.

9. Let Q be the open square with corners at $1+i$, $-1+i$, $-1-i$, and $1-i$; i.e., $Q = \{z \in \mathbf{C} : |\Re(z)| < 1\} \cap \{z \in \mathbf{C} : |\Im(z)| < 1\}$. Suppose that $f \in H(Q)$ maps Q one-to-one and onto D, and also satisfies $f(0) = 0$. Show that, for all $z \in Q$, $f(iz) = if(z)$.

Solution. We can find a $c \in \mathbf{C}$ with $|c| = 1$ such that $\tilde{f} \equiv cf$ satisfies $\tilde{f}'(0) > 0$. Since $z \mapsto cz$ takes D one-to-one and onto itself, $\tilde{f}$ will also map Q one-to-one and onto D. Also, $\tilde{f}(0) = 0$. Therefore $\tilde{f}$ is the unique Riemann mapping (for the point $0 \in Q$) guaranteed by the Riemann Mapping Theorem. Define $g(z) \equiv -i\tilde{f}(iz)$. Since $z \mapsto -iz$ sends D to itself and $z \mapsto iz$ sends Q to itself (one-to-one and onto, of course), g maps Q one-to-one and onto D. Clearly $g(0) = 0$; but also, by the chain rule,

$$g'(0) = -i(i)\tilde{f}'(0) = \tilde{f}'(0) > 0,$$

which, by the uniqueness of the Riemann mapping, implies $g = \tilde{f}$. I.e.,

$$-icf(iz) = cf(z),$$

or, after dividing,

$$f(iz) = if(z).$$

10. Suppose that $f : D \mapsto D$ is analytic and has zeroes at the points $a_1, a_2, \ldots, a_n$ in D. Show that, for all $z \in D$,

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - a_k}{1 - z\bar{a_k}} \right|.$$

Hint: If $\alpha \in D$, what is the absolute value of

$$\frac{z - \alpha}{1 - z\bar{\alpha}}$$

when $|z| = 1$?

Solution. The function

$$g(z) \equiv \frac{f(z)}{\prod_{k=1}^{n} \frac{z - a_k}{1 - z\bar{a_k}}}$$

has removable singularities at the a_n's. Assume they've been filled in, so that $g \in H(D)$. Given $\epsilon > 0$, let $r < 1$ be so close to 1 that

$$\prod_{k=1}^{n} \left| \frac{z - a_k}{1 - z\bar{a_k}} \right| > 1 - \epsilon$$

everywhere on $\{z : \ |z| = r\}$. Since $|f| < 1$ in all of D, the Maximum Principle now implies that

$$|g| \leq \left(\prod_{k=1}^{n} \left| \frac{z - a_k}{1 - z\bar{a_k}} \right| \right)^{-1} \leq (1 - \epsilon)^{-1}$$

everywhere in $\Delta(0; r)$. Letting $r \to 1^-$, we get that $|g| \leq (1 - \epsilon)^{-1}$ in all of D. But ϵ is arbitrary; therefore $|g| \leq 1$ in all of D; implying that

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - a_k}{1 - z\bar{a_k}} \right|$$

whenever $z \notin \{a_1, \ldots, a_n\}$; of course, the inequality is trivial when $z \in \{a_1, \ldots, a_n\}$. Done.