An Important Lemma.

Suppose that $\Omega \subset \mathbf{C}$ is open, $f : \Omega \mapsto \mathbf{C}$ is analytic, and we define $\phi : \Omega \times \Omega \mapsto \mathbf{C}$ by

$$\phi(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w; \\ f'(z) & \text{if } z = w. \end{cases}$$

Then ϕ is continuous on $\Omega \times \Omega$ and, for each $(z_0, w_0) \in \Omega \times \Omega$, the functions $\phi(z, w_0) : \Omega \mapsto \mathbf{C}$ and $\phi(z_0, w) : \Omega \mapsto \mathbf{C}$ are analytic (in z and w, respectively).

Proof. We show analyticity first. Since $\phi(z,w) = \phi(w,z)$, it suffices to show that $\phi(z, w_0)$ is analytic in z for all w_0. Set $h(z) = f(z) - f(w_0)$. Note that $h(w_0) = 0$. By a theorem from class, we can write $f(z) - f(w_0) = h(z) = (z - w_0)g(z)$, where g is analytic on all of Ω. The definition of g implies that $g(w_0) = f'(w_0)$. Therefore $\phi(z, w_0) = g(z)$, which is analytic in z.

Showing continuity amounts to showing that if $z_n \to z \in \Omega$ and $w_n \to w \in \Omega$ then $\phi(z_n, w_n) \to \phi(z,w)$. This is trivial if $z \neq w$. Therefore it is enough to show the following: If $a \in \Omega$, $z_n \to a$, and $w_n \to a$, then $\phi(z_n, w_n) \to \phi(a,a) = f'(a)$. To simplify things (but without loss of generality), we'll assume $a = 0$. Let $r > 0$ be such that $\Delta(0; r) \subset \Omega$. Now write f as a power series around 0,

$$f(z) = \sum_0^\infty \alpha_k z^k,$$

which has r.o.c. $\geq r$. For $k \geq 1$ define

$$\psi_k(z,w) = \sum_0^{k-1} z^j w^{k-1-j},$$

where we set $\psi_1(z,w) = 1$. The important thing to notice is that $z^k - w^k = (z-w)\psi_k(z,w)$, while $\psi_k(z,z) = kz^{k-1} = (z^k)'$. Therefore, if $z \neq w$, and z and w both lie in $\Delta(0; r)$,

$$\phi(z,w) = \frac{f(z) - f(w)}{z - w} = \sum_1^\infty \alpha_k \psi_k(z,w);$$

while, if $z = w \in \Delta(0; r)$,

$$\phi(z,w) = f'(z) = \sum_1^\infty k\alpha_k z^{k-1} = \sum_1^\infty \alpha_k \psi_k(z,w).$$

Upshot:

$$\phi(z,w) = \sum_{1}^{\infty} \alpha_k \psi_k(z,w)$$

when z and w lie in $\Delta(0;r)$. Let $0 < \rho_1 < \rho_2 < r$. Our earlier work on power series implies that $k\alpha_k\rho_2^{k-1} \to 0$ as $k \to \infty$, and thus there is a B such that $|k\alpha_k\rho_2^{k-1}| \le B$ for all k. If z and w are both in $\Delta(0;\rho_1)$ then $|\alpha_k\psi_k(z,w)| \le |k\alpha_k\rho_1^{k-1}| \le B(\rho_1/\rho_2)^{k-1} \equiv B\eta^{k-1}$, where $0 < \eta < 1$. This implies that the series $\sum_1^\infty \alpha_k\psi_k(z,w)$ satisfies the hypotheses of the Weierstrass M-test inside $\Delta(0;\rho_1)$. However, it is trivial that each $\psi_k(z,w)$ is continuous at $(0,0)$. Therefore the infinite series is too, and $\phi(z_n,w_n) \to \phi(0,0)$. QED.