Math 331 Midterm Exam Solutions **Spring, 2010**

The three numbered problems are worth 10 points each, for a total of 30. The two extra credit problems are worth 12 points each. You may use without proof any facts from freshman calculus or Math 241 (e.g., L'Hospital's Rule, the values of familiar limits and integrals).

1. If (X, d) is a metric space, a subset $E \subset X$ is called *discrete* if there is a $\delta > 0$ so that, for all points x and y in E, if $x \neq y$, then $d(x, y) > \delta$. Show that every discrete subset is closed.

Solution (one of many possible). Suppose $\{x_n\} \subset E$ is a sequence that converges to p. There exists an N such that $\forall m,\, n \geq N\,(d(x_m, x_n) < \delta)$. E's discreteness now implies that $x_n = x_N$ for all $n \geq N$. Therefore $p = x_N \in E$, and E is closed.

2. Find, with justifications, the radii of convergence of:
a)

$$\sum_1^\infty n^n z^n.$$

b)

$$\sum_1^\infty n^n z^{n^2}.$$

Solutions. a) Apply the Root Test: $(n^n)^{1/n} = n \to \infty$, and the radius is 0. b) Same test, but now $(n^n)^{1/n^2} = n^{1/n} \to 1$, and the radius is 1.

3. Suppose that $a \in \mathbf{C}$ and $r > 0$, and let $f : \Delta(a; r) \mapsto \mathbf{C}$ be analytic. Suppose that, for all $z \in \Delta(a; r)$,

$$|f(z)| \geq \sqrt{|z - a|}.$$

Show that $f(a) \neq 0$.

Solution. If $f(a) = 0$ there is an analytic $g : \Delta(a; r) \mapsto \mathbf{C}$ such that $f(z) = (z - a)g(z)$ on all of the disc. If $z \neq a$ then $|f(z)| \geq \sqrt{|z - a|}$ implies that $|g(z)| \geq |z - a|^{-1/2}$, which implies that $|g(z)| \to \infty$ as $z \to a$, a contradiction. By the way, you don't actually need analyticity for this!

Extra Credit 1. Let (X, d) be a metric space, and suppose that $\{E_\alpha\}_\alpha$ is familiy (possibly uncountable) of *closed* subsets of X. Suppose that there is a $\delta > 0$ such that, if $\alpha \neq \alpha'$, $x \in E_\alpha$, and $y \in E_{\alpha'}$, then $d(x, y) > \delta$. Show that the union $\cup_\alpha E_\alpha$ is a closed subset of X.

Solution. Set $S = \cup_\alpha E_\alpha$. Let $\{x_n\} \subset S$ be a sequence such that $x_n \to p$. There exists an N such that $\forall m,\, n \geq N\,(d(x_m, x_n) < \delta)$, which means that there is an α such that, for all $n \geq N$, $x_n \in E_\alpha$. But E_α is closed; therefore p (the limit of the sequence) lies in $E_\alpha \subset S$, implying S is closed.

Extra Credit 2. Find, with justification, the radius of convergence of

$$\sum_{1}^{\infty} \frac{n^n}{n!} z^n.$$

Solution. Set $a_n = n^n/n!$ and apply the Ratio Test.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}/n^n}{(n+1)!/n!} = \left(1 + \frac{1}{n}\right)^n,$$

which converges to e. The radius is $1/e$.