

Stability of wavelet-like expansions under chromatic aberration

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Abstract.

We prove that wavelet and wavelet-like expansions of functions are L^p -stable under small (but otherwise arbitrary and independent) errors in translation and dilation of the constituent reproducing kernels. These perturbations are frequency-dependent, which is why we call them “chromatic aberration.” We show that, if these errors have sizes no bigger than η , then the L^p distance between the “true” and “perturbed” output functions is bounded by a constant times $\eta^\tau \|f\|_p$, where τ is a positive number depending on the family of kernels in question. We show that this result also holds in $L^p(w)$ if w is a Muckenhoupt A_p weight.

1. Introduction.

In this paper we let \mathcal{D} denote the collection of dyadic cubes in \mathbf{R}^d . We suppose that we have two families of functions, $\{\phi^{(Q)}\}_{Q \in \mathcal{D}}$ and $\{\psi^{(Q)}\}_{Q \in \mathcal{D}}$, both indexed over \mathcal{D} . The functions $\phi^{(Q)}$ and $\psi^{(Q)}$ satisfy some uniform decay, cancelation, and smoothness conditions that make them “wavelet-like.” To wit: if α and ϵ are positive numbers, we shall say that $h : \mathbf{R}^d \mapsto \mathbf{R}$ belongs to $\mathcal{C}_{(\alpha, \epsilon)}$ if it satisfies the following three conditions:

1. For all $x \in \mathbf{R}^d$,

$$|h(x)| \leq (1 + |x|)^{-d-\epsilon}.$$

2. For all x and x' in \mathbf{R}^d ,

$$|h(x) - h(x')| \leq |x - x'|^\alpha \left((1 + |x|)^{-d-\epsilon} + (1 + |x'|)^{-d-\epsilon} \right).$$

- 3.

$$\int_{\mathbf{R}^d} h(x) dx = 0.$$

We will assume that, for some **fixed** α and ϵ , every $\phi^{(Q)}$ and $\psi^{(Q)}$ belongs to $\mathcal{C}_{(\alpha, \epsilon)}$.

If Q is any cube, we use x_Q to denote its center, $\ell(Q)$ for its sidelength, and $|Q|$ for its Lebesgue measure (we shall also use $|\cdot|$ to denote the Lebesgue measures of other kinds of sets); these are all standard notations. If $h \in \mathcal{C}_{(\alpha, \epsilon)}$, we use $h_{(Q)}$ to mean

$$|Q|^{-1/2} h((x - x_Q)/\ell(Q)).$$

In other words, where h was “centered” around 0, $h_{(Q)}$ is centered around x_Q , and $h_{(Q)}$ has been re-scaled to the dimensions of Q in a way that preserves h 's L^2 norm.

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If \mathcal{F} is any finite subset of \mathcal{D} , the following expression is defined for any locally integrable f with reasonable decay:

$$T(f) \equiv \sum_{Q \in \mathcal{F}} \langle f, \phi_{(Q)}^{(Q)} \rangle \psi_{(Q)}^{(Q)}. \quad (1.1)$$

(We apologize for the awkward notation; it will soon get worse.) By “locally integrable with reasonable decay,” we mean that $|f|(1 + |x|)^{-d-\epsilon}$ belongs to L^1 . We are using $\langle \cdot, \cdot \rangle$ to denote the usual inner product between functions defined on \mathbf{R}^d :

$$\langle f, g \rangle \equiv \int_{\mathbf{R}^d} f(x) \overline{g(x)} dx.$$

It is well known that (1.1) defines a bounded operator on $L^p(w)$ for any $1 < p < \infty$ and any weight w in the Muckenhoupt A_p class (see the arguments in [W], chapter 7, and exercise 6.10 in the previous chapter):

$$\left\| \sum_{Q \in \mathcal{F}} \langle f, \phi_{(Q)}^{(Q)} \rangle \psi_{(Q)}^{(Q)} \right\|_{L^p(w)} \leq C \|f\|_{L^p(w)}, \quad (1.2)$$

where the constant C depends on α , ϵ , p , and w , but not on the family \mathcal{F} . Let us recall that a non-negative $w \in L^1_{loc}(\mathbf{R}^d)$ belongs to A_p ($1 < p < \infty$) if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right)^{1/p} \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{1/p'} < \infty,$$

where the supremum is over all bounded cubes $Q \subset \mathbf{R}^d$, and p' is p 's dual exponent. We refer the reader to any of [D], [G], [GCRdF], or [St] for masterful introductions to the theory of A_p weights.

Inequality (1.2)—with the help of Littlewood-Paley theory—lets us make sense of

$$\sum_{Q \in \mathcal{D}} \langle f, \phi_{(Q)}^{(Q)} \rangle \psi_{(Q)}^{(Q)}$$

as a bounded operator on $L^p(w)$. We are restricting ourselves to finite sums only because we don't want to bother about technical problems of convergence.

The title of this paper refers to the extent to which (1.1) is stable under “tweaking” of the families $\{\phi_{(Q)}^{(Q)}\}_{Q \in \mathcal{D}}$ and $\{\psi_{(Q)}^{(Q)}\}_{Q \in \mathcal{D}}$. More precisely, we are interested in what happens when the families of kernels $\{\phi_{(Q)}^{(Q)}\}_{Q \in \mathcal{D}}$ and $\{\psi_{(Q)}^{(Q)}\}_{Q \in \mathcal{D}}$ get shifted by small translations and changes of scale. (This shifting is the “chromatic aberration” of the title; we will try to justify it shortly.) To be precise, we suppose that we have a fixed number η , with

$0 \leq \eta \leq 1/2$, and that, for every $Q \in \mathcal{D}$, we have fixed vectors s_Q and t_Q in \mathbf{R}^d and fixed numbers y_Q and z_Q . We assume that these vectors and numbers satisfy

$$|s_Q| \leq \eta \tag{1.3}$$

$$|t_Q| \leq \eta \tag{1.4}$$

$$|1 - y_Q| \leq \eta \tag{1.5}$$

$$|1 - z_Q| \leq \eta \tag{1.6}$$

for every Q , but that they are otherwise arbitrary. For every $Q \in \mathcal{D}$ we define

$$\begin{aligned} \tilde{\phi}^{(Q)}(x) &\equiv (y_Q)^{-d} \phi^{(Q)}((x - s_Q)/y_Q) \\ \tilde{\psi}^{(Q)}(x) &\equiv (z_Q)^{-d} \psi^{(Q)}((x - t_Q)/z_Q). \end{aligned}$$

In plain language, $\tilde{\phi}^{(Q)}$ (respectively, $\tilde{\psi}^{(Q)}$) is obtained from $\phi^{(Q)}$ (respectively, $\psi^{(Q)}$) by translating it by a small amount and by dilating it slightly.

Corresponding to these “tweaked” families, we have a tweaked linear operator $\tilde{T}(f)$:

$$\tilde{T}(f) \equiv \sum_{Q \in \mathcal{F}} \langle f, \tilde{\phi}_{(Q)}^{(Q)} \rangle \tilde{\psi}_{(Q)}^{(Q)}.$$

The functions $T(f)$ and $\tilde{T}(f)$ should not be very far apart if η is small. The main result of this paper, which we prove in section 3, makes this intuitively appealing statement precise.

Theorem 1.1. *Let \mathcal{F} , $\{\phi^{(Q)}\}_{Q \in \mathcal{D}}$, and $\{\psi^{(Q)}\}_{Q \in \mathcal{D}}$ be as described above. There is a $\tau > 0$, with an allowable range depending only on α , ϵ , and d ; and, if $w \in A_p$ ($1 < p < \infty$), there is a C , depending only on τ , α , ϵ , p , and w , such that, if $0 \leq \eta \leq 1/2$ and $\{s_Q\}_{Q \in \mathcal{D}}$, $\{t_Q\}_{Q \in \mathcal{D}}$, $\{y_Q\}_{Q \in \mathcal{D}}$, $\{z_Q\}_{Q \in \mathcal{D}}$, satisfy (1.3)–(1.6), then, for all $f \in L^p(w)$,*

$$\|T(f) - \tilde{T}(f)\|_{L^p(w)} \leq C\eta^\tau \|f\|_{L^p(w)}.$$

In other words, operators like $T(\cdot)$ are stable—and vary, as measured by L^p norms, in a Hölder continuous manner—with respect to small translations and dilations. We have likened these perturbations to chromatic aberration. One can think of $T(\cdot)$ as modeling the *ideal* action of a black box (which might be a lens) on a signal. In reality, bands of f ’s spectrum will get shifted or dilated away from the ideal in ways depending on their frequency and spatial position. We model this deviation by replacing each term

$$\langle f, \phi_{(Q)}^{(Q)} \rangle \psi_{(Q)}^{(Q)}$$

with its perturbed version,

$$\langle f, \tilde{\phi}_{(Q)}^{(Q)} \rangle \tilde{\psi}_{(Q)}^{(Q)}.$$

Theorem 1.1 says that, as long as the shifts and dilations are bounded by a constant times the corresponding wavelengths, the real action of the black box stays very close to the ideal.

We introduce some definitions, recall some facts about the intrinsic square function (see [W]), and prove one technical lemma in section 2. We prove Theorem 1.1 in section 3.

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2. Preliminaries.

Definition 2.1. *Let β and δ be positive numbers. If $|f|(1 + |x|)^{-d-\delta} \in L^1(\mathbf{R}^d, dx)$ and $(t, y) \in \mathbf{R}_+^{d+1} \equiv \mathbf{R}^d \times (0, \infty)$, we set*

$$\tilde{A}_{(\beta, \delta)}(f)(t, y) \equiv \sup\{|f * \phi_y(t)| : \phi \in \mathcal{C}_{(\beta, \delta)}\},$$

where $\phi_y(x) \equiv y^{-d}\phi(x/y)$, the usual L^1 -dilation. For $x \in \mathbf{R}^d$, we define

$$\tilde{G}_{(\beta, \delta)}(f)(x) \equiv \left(\int_{(t, y): |x-t| < y} (\tilde{A}_{(\beta, \delta)}(f)(t, y))^2 \frac{dt dy}{y^{d+1}} \right)^{1/2},$$

the intrinsic square function of f , of order (β, δ) .

The intrinsic square function is a sort of “grand maximal” square function. If β and δ are small enough, it dominates almost all of the classical square functions¹. On the other hand, it is not, on the average, bigger than any of them, as shown by the following theorem (proved in [W], Theorem 7.2).

Theorem 2.1. *Let β and δ be positive numbers and suppose that $1 < p < \infty$. If w is an A_p weight, there is a constant C , depending only on β , δ , p , and w , such that, for all $f \in L^p(w)$,*

$$\|\tilde{G}_{(\beta, \delta)}(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

The definition of the intrinsic square function is based on a family of functions— $\mathcal{C}_{(\beta, \delta)}$ —which is normalized in a certain way. It will be convenient for us to use this normalization to build an actual normed space.

Definition 2.2. *Let β and δ be positive numbers. We let $N(\beta, \delta)$ be the space of functions ϕ such that, for some $\lambda > 0$, $\phi/\lambda \in \mathcal{C}_{(\beta, \delta)}$. We put a norm, $\|\cdot\|_{(\beta, \delta)}$, on $N(\beta, \delta)$ by setting*

$$\|\phi\|_{(\beta, \delta)} \equiv \inf\{\lambda > 0 : \phi/\lambda \in \mathcal{C}_{(\beta, \delta)}\}.$$

The reader should note that the preceding definition turns $\mathcal{C}_{(\beta, \delta)}$ into $N(\beta, \delta)$'s unit ball: if we wished, we could rewrite the definition of $\tilde{A}_{(\beta, \delta)}(f)(t, y)$ as

$$\tilde{A}_{(\beta, \delta)}(f)(t, y) \equiv \sup\{|f * \phi_y(t)| : \phi \in N(\beta, \delta), \|\phi\|_{(\beta, \delta)} \leq 1\}.$$

¹ It doesn't dominate g_λ^* , but in many cases it makes g_λ^* unnecessary

Suppose that, for $R > 0$, we were to define

$$\tilde{A}_{(\beta,\delta),R}(f)(t,y) \equiv \sup\{|f * \phi_y(t)| : \phi \in N(\beta,\delta), \|\phi\|_{(\beta,\delta)} \leq R\};$$

and, analogously,

$$\tilde{G}_{(\beta,\delta),R}(f)(x) \equiv \left(\int_{(t,y): |x-t|<y} (\tilde{A}_{(\beta,\delta),R}(f)(t,y))^2 \frac{dt dy}{y^{d+1}} \right)^{1/2}.$$

Then it is obvious, but well worth noting, that

$$\tilde{G}_{(\beta,\delta),R}(f)(x) = R\tilde{G}_{(\beta,\delta)}(f)(x) \tag{2.1}$$

pointwise. The following observation will also be useful: If $\{\rho^{(Q)}\}_{Q \in \mathcal{D}}$ is any family in $\mathcal{C}_{(\beta,\delta)}$ then, for any f for which $\tilde{G}_{(\beta,\delta)}(f)$ makes sense,

$$\left(\sum_{Q \in \mathcal{D}} \frac{|\langle f, \rho^{(Q)} \rangle|^2}{|Q|} \chi_Q \right)^{1/2} \leq C(\beta, \delta, d) \tilde{G}_{(\beta,\delta)}(f) \tag{2.2}$$

pointwise. Inequality (2.2) follows from the fact that, if Q is any dyadic cube, $x \in Q$, and $\ell(Q) = 2^k$, then

$$\frac{|\langle f, \rho^{(Q)} \rangle|^2}{|Q|} \leq C(\beta, \delta, d) \int_{(t,y): |x-t|<y, 2^k < y \leq 2^{k+1}} (\tilde{A}_{(\beta,\delta)}(f)(t,y))^2 \frac{dt dy}{y^{d+1}}.$$

Theorem 1.1 will be an easy consequence of the preceding observations and the following lemma. The lemma is the only non-trivial thing we will actually prove in this paper.

Lemma 2.1. *Let β and δ be positive numbers. There exist numbers $0 < \beta' < \beta$, $0 < \delta' < \delta$, $\tau > 0$, and a positive constant C such that, if $0 \leq \eta \leq 1/2$, $|t| \leq \eta$ ($t \in \mathbf{R}^d$), $|1 - y| \leq \eta$, and $\phi \in N(\beta, \delta)$, then*

$$\|\phi - \phi_{(t,y)}\|_{(\beta',\delta')} \leq C\eta^\tau \|\phi\|_{(\beta,\delta)},$$

where $\phi_{(t,y)}(x)$ is defined to be $y^{-d}\phi((x-t)/y)$.

The lemma calls for some explanation. We think of $\phi_{(t,y)}$ as a perturbed version of ϕ : it's been shifted by t (which is small) and dilated by y (which is close to 1). The lemma says that, if $\phi \in N(\beta, \delta)$, then the distance between ϕ and $\phi_{(t,y)}$, as measured by the $N(\beta', \delta')$ norm, will be bounded by a constant times $(|t| + |1 - y|)^\tau \|\phi\|_{(\beta,\delta)}$. Such a bound is not as strange as it might look. For $\alpha > 0$, define $Lip(\alpha)$ to be the space of functions $f : \mathbf{R} \mapsto \mathbf{R}$ such that

$$\|f\|_{(\alpha)} \equiv \sup_{x \in \mathbf{R}} |f(x)| + \sup_{\substack{x,y \in \mathbf{R} \\ 0 < |x-y| \leq 1}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. If $f \in Lip(1)$ and $|t| \leq 1$, there is no reason to expect $f(\cdot) - f(\cdot - t)$ to have a small $Lip(1)$ norm. However, it is not hard to show that, for any $0 < \alpha < 1$,

$$\|f(\cdot) - f(\cdot - t)\|_{(\alpha)} \leq 3|t|^{1-\alpha}\|f\|_{(1)}. \quad (2.3)$$

Here is a quick proof of (2.3). Assume $\|f\|_{(1)} \leq 1$. For any $x \in \mathbf{R}$,

$$|f(x) - f(x - t)| \leq |t| \leq |t|^{1-\alpha},$$

because $|t| \leq 1$. Now let x and y satisfy $0 < |x - y| \leq 1$. If $|x - y| > |t|$ then

$$\begin{aligned} |f(x) - f(x - t) - f(y) - f(y - t)| &\leq 2|t| \\ &= 2|t|^{1-\alpha}|t|^\alpha \\ &\leq 2|t|^{1-\alpha}|x - y|^\alpha. \end{aligned}$$

If $|x - y| \leq |t|$ then

$$\begin{aligned} |f(x) - f(x - t) - f(y) - f(y - t)| &\leq 2|x - y| \\ &= 2|x - y|^\alpha|x - y|^{1-\alpha} \\ &\leq 2|x - y|^\alpha|t|^{1-\alpha}. \end{aligned}$$

Combining the 3 inequalities yields (2.3). The corresponding arguments for $N(\beta, \delta)$ are complicated by the fact that there we also have to bound the rates of decay of a function and its Hölder modulus, but the essential ideal is like that behind (2.3).

Proof of Lemma 2.1. Without loss of generality, we assume that $\|\phi\|_{(\beta, \delta)} = 1$; i.e., that $\phi \in \mathcal{C}_{(\beta, \delta)}$. Let us note that, for any $0 < \tilde{\beta} \leq \beta$ and $0 < \tilde{\delta} \leq \delta$, we have the inclusion $\mathcal{C}_{(\beta, \delta)} \subset \mathcal{C}_{(\tilde{\beta}, \tilde{\delta})}$. The decay implication is easy to see, because, for any $x \in \mathbf{R}$,

$$(1 + |x|)^{-d-\delta} \leq (1 + |x|)^{-d-\tilde{\delta}}.$$

The Hölder modulus condition is a little trickier. It rests on the fact that, if ϕ satisfies

$$|\phi(x)| \leq (1 + |x|)^{-d-\delta}$$

for all x , then it will satisfy

$$|\phi(x) - \phi(x')| \leq |x - x'|^\beta((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}) \quad (2.4)$$

for all x and x' , if and only if it satisfies (2.4) for all x and x' such that $|x - x'| \leq 1$. But then, if $|x - x'| \leq 1$, and ϕ satisfies (2.4), it is trivial that ϕ satisfies

$$|\phi(x) - \phi(x')| \leq |x - x'|^{\tilde{\beta}}((1 + |x|)^{-d-\tilde{\delta}} + (1 + |x'|)^{-d-\tilde{\delta}}),$$

because $\tilde{\beta} \leq \beta$ and $\tilde{\delta} \leq \delta$.

Therefore, without loss of generality, we can assume that $\beta < \delta/2$.

Recall that $\phi_{(t,y)}(x) = y^{-d}\phi((x/y) - (t/y))$. We write:

$$\phi(x) - \phi_{(t,y)}(x) = (1 - y^{-d})\phi(x) + y^{-d}(\phi(x) - \phi(x/y)) + y^{-d}(\phi(x/y) - \phi((x/y) - (t/y))).$$

But $|1 - y^{-d}| \leq C\eta$, $y^{-d} \leq C$, and $|t/y| \leq 2\eta$. Therefore it will be enough to show that, if $|1 - y| \leq \eta \leq 1/2$ and $|t| \leq \eta \leq 1$ then

$$\|\phi(\cdot) - \phi(\cdot/y)\|_{(\beta', \delta')} \leq C\eta^\tau \quad (2.5)$$

and

$$\|\phi(\cdot) - \phi(\cdot - t)\|_{(\beta', \delta')} \leq C\eta^\tau, \quad (2.6)$$

for appropriate β' , δ' , and τ .

It is in proving (2.5) that we will use our assumption that $\beta < \delta/2$. As the reader will see, it would have been okay to simply have $\beta < \delta$, but we believe that specifying $\beta < \delta/2$ makes the argument easier to understand.

Proof of (2.5). Write $x/y = x + (y^{-1} - 1)x$, and note that $|y^{-1} - 1| \leq C\eta$. Therefore, for any x ,

$$\begin{aligned} |\phi(x) - \phi(x/y)| &\leq |(y^{-1} - 1)x|^\beta ((1 + |x|)^{-d-\delta} + (1 + |x/y|)^{-d-\delta}) \\ &\leq C\eta^\beta |x|^\beta (1 + |x|)^{-d-\delta} \\ &\leq C\eta^\beta (1 + |x|)^\beta (1 + |x|)^{-d-\delta} \\ &\leq C\eta^\beta (1 + |x|)^{-d-\delta/2}, \end{aligned}$$

where the last inequality is true because $\beta < \delta/2$. Therefore $\phi(\cdot) - \phi(\cdot/y)$ has the right amount of decay at infinity.

Let x and x' belong to \mathbf{R}^d . If $|x - x'| \geq \eta$ then, from the preceding estimate,

$$\begin{aligned} &|(\phi(x) - \phi(x/y)) - (\phi(x') - \phi(x'/y))| \\ &\leq |\phi(x) - \phi(x/y)| + |\phi(x') - \phi(x'/y)| \\ &\leq C\eta^\beta ((1 + |x|)^{-d-\delta/2} + (1 + |x'|)^{-d-\delta/2}) \\ &\leq C\eta^{\beta/2} |x - x'|^{\beta/2} ((1 + |x|)^{-d-\delta/2} + (1 + |x'|)^{-d-\delta/2}), \end{aligned}$$

which is fine. If $|x - x'| \leq \eta$ then

$$\begin{aligned} &|(\phi(x) - \phi(x/y)) - (\phi(x') - \phi(x'/y))| \\ &\leq |\phi(x) - \phi(x')| + |\phi(x/y) - \phi(x'/y)| \\ &\leq C|x - x'|^\beta ((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}) \\ &\leq C\eta^{\beta/2} |x - x'|^{\beta/2} ((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}) \\ &\leq C\eta^{\beta/2} |x - x'|^{\beta/2} ((1 + |x|)^{-d-\delta/2} + (1 + |x'|)^{-d-\delta/2}), \end{aligned}$$

which is also fine.

For the reader who is keeping score, we have just shown that (2.5) is true for $\tau = \beta' = \beta/2$ and $\delta' = \delta/2$, under the assumption that $\beta < \delta/2$.

Proof of (2.6). For any x ,

$$|\phi(x) - \phi(x - t)| \leq C\eta^\beta(1 + |x|)^{-d-\delta},$$

because $|t| \leq \eta \leq 1$. This shows that $\phi(\cdot) - \phi(\cdot - t)$ has enough decay.

Let x and x' belong to \mathbf{R}^d . If $|x - x'| \geq \eta$ then, arguing as we did regarding (2.5),

$$\begin{aligned} & |(\phi(x) - \phi(x - t)) - (\phi(x') - \phi(x' - t))| \\ & \leq |\phi(x) - \phi(x - t)| + |\phi(x') - \phi(x' - t)| \\ & \leq C\eta^\beta((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}) \\ & \leq C\eta^{\beta/2}|x - x'|^{\beta/2}((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}), \end{aligned}$$

which is what we want. On the other hand, if $|x - x'| \leq \eta$,

$$\begin{aligned} & |(\phi(x) - \phi(x - t)) - (\phi(x') - \phi(x' - t))| \\ & \leq |\phi(x) - \phi(x')| + |\phi(x - t) - \phi(x' - t)| \\ & \leq C|x - x'|^\beta((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}), \end{aligned} \tag{2.7}$$

where we are using the fact that $(1 + |x|)$ and $(1 + |x - t|)$ are comparable, independent of x , because $|t| \leq \eta \leq 1$. Continuing, the right-hand side of (2.7) is less than or equal to

$$C\eta^{\beta/2}|x - x'|^{\beta/2}((1 + |x|)^{-d-\delta} + (1 + |x'|)^{-d-\delta}), \tag{2.8}$$

which finishes the proof of (2.6) and of the lemma. Inequality (2.8) is consistent with the remark we made after the proof of (2.5). Under the (harmless) assumption that $\beta < \delta/2$, Lemma 2.1 holds for $\tau = \beta' = \beta/2$ and $\delta' = \delta/2$.

3. Proof of Theorem 1.1.

We can write $T(f) - \tilde{T}(f)$ as

$$\sum_{Q \in \mathcal{F}} \langle f, \phi_{(Q)}^{(Q)} - \tilde{\phi}_{(Q)}^{(Q)} \rangle \psi_{(Q)}^{(Q)} + \sum_{Q \in \mathcal{F}} \langle f, \tilde{\phi}_{(Q)}^{(Q)} \rangle (\psi_{(Q)}^{(Q)} - \tilde{\psi}_{(Q)}^{(Q)}) \equiv (I) + (II).$$

By an easy and well-known duality argument (see [W], chapter 7), it suffices to bound (I). By standard Littlewood-Paley theory, the $L^p(w)$ norm of (I) is bounded by a constant times

$$\left\| \left(\sum_{Q \in \mathcal{F}} \frac{|\langle f, \phi_{(Q)}^{(Q)} - \tilde{\phi}_{(Q)}^{(Q)} \rangle|^2}{|Q|} \chi_Q \right)^{1/2} \right\|_{L^p(w)}.$$

For each Q , set $\rho^{(Q)} = \phi^{(Q)} - \tilde{\phi}^{(Q)}$. Because of Lemma 2.1, each $\rho^{(Q)}$ satisfies

$$\|\rho^{(Q)}\|_{(\beta,\delta)} \leq C\eta^\tau,$$

for some C , for appropriate β , δ , and τ . When we combine this with (2.1) and (2.2), we get that

$$\left(\sum_{Q \in \mathcal{F}} \frac{|\langle f, \phi_{(Q)}^{(Q)} - \tilde{\phi}_{(Q)}^{(Q)} \rangle|^2}{|Q|} \chi_Q \right)^{1/2} \leq C\eta^\tau \tilde{G}_{(\beta,\delta)}(f)$$

pointwise. But Theorem 2.1 says that

$$\|\tilde{G}_{(\beta,\delta)}(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

Theorem 1.1 is proved.

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